Big Ideas Flow Vertically Without Limits

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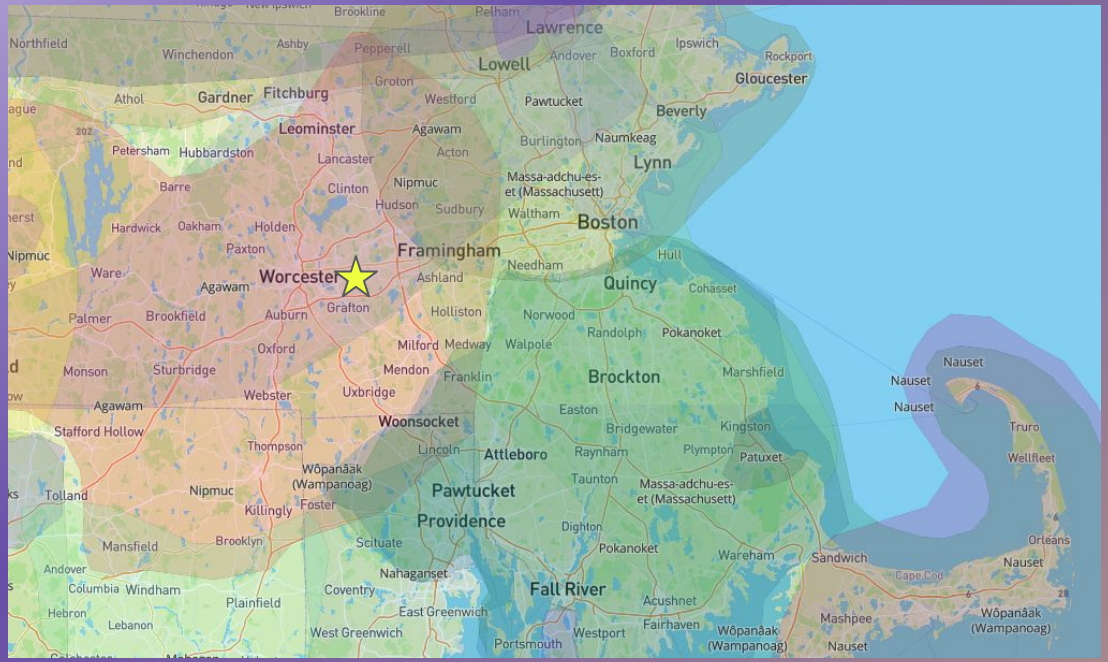
Dr. Paul Gray
NCSM Past President
Cosenza & Associates, LLC

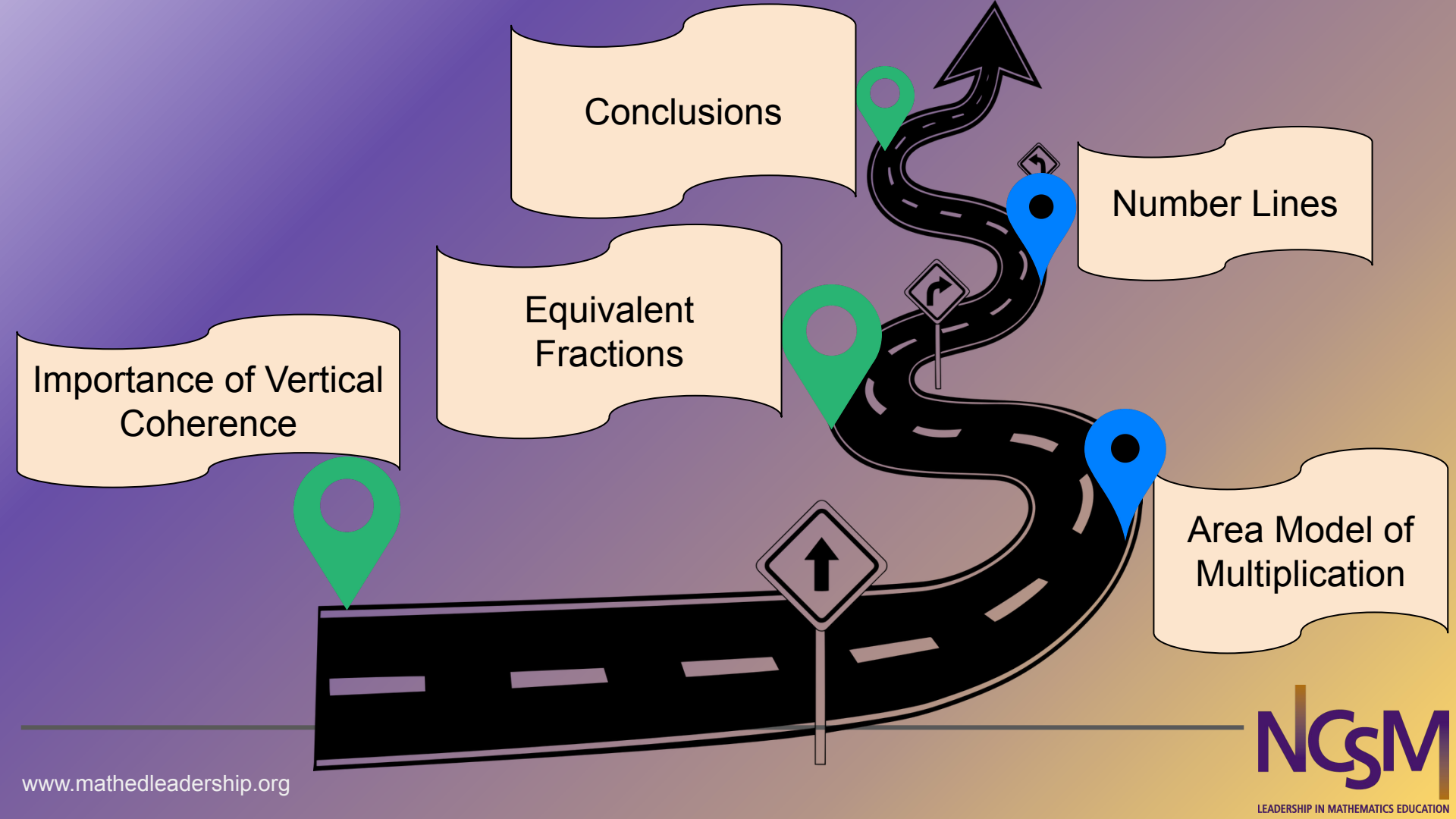
NCSM
LEADERSHIP IN MATHEMATICS EDUCATION



Ancestral lands of the Agawam people Nipmuc Nation

Source: https://native-land.ca/

Learn more: https://www.nipmucnation.org/



Vertical Coherence

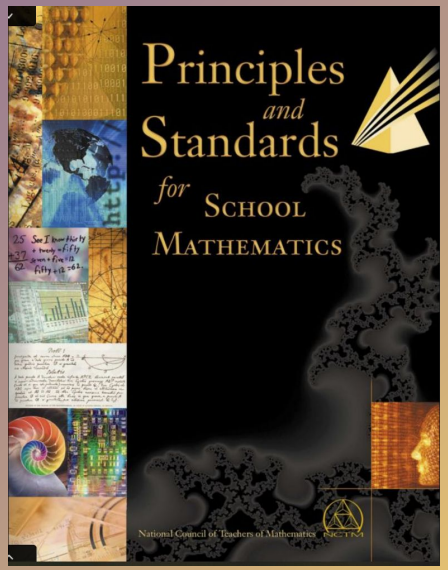
A coherent curriculum effectively organizes and integrates important mathematical ideas so that students can see how the ideas build on, or connect with, other ideas, thus enabling them to develop new understandings and skills.

NCTM, *Principles and Standards for School Mathematics*, 2000.

Principles and Standards for School Mathematics

Curriculum Principle

- A mathematics curriculum should be coherent.
- A mathematics curriculum should focus on important mathematics.
- A mathematics curriculum should be well articulated across the grades.



NCTM, 2000

Curriculum Focal Points

What are Curriculum Focal Points?

- Important mathematical topics for each grade level
- Core structures that lay a conceptual foundation and serve to organize content, connecting and bringing coherence to multiple concepts and processes taught at and across grade levels

NCTM, 2006

Common Core State Standards for Mathematics

What does coherence look like?

To be coherent, a set of content standards must evolve from particulars (e.g., the meaning and operations of whole numbers, including simple math facts and routine computational procedures associated with whole numbers and fractions) to deeper structures inherent in the discipline. These deeper structures then serve as a means for connecting the particulars (such as an understanding of the rational number system and its properties).

CCSSO, 2010

Schmidt & Houang, 2002, cited in *CCSSM*, 2010, p. 4

Progression of Pre-K–8 Domains										
Domain	**Grade Level**									
	PK	**K**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Counting and Cardinality	■	■								
Operations and Algebraic Thinking	■	■	■	■	■	■	■			
Number and Operations in Base Ten		■	■	■	■	■	■			
Number and Operations – Fractions					■	■	■			
The Number System								■	■	■
Ratios and Proportional Relationships								■	■	
Expressions and Equations								■	■	■
Functions										■
Measurement and Data	■	■	■	■	■	■	■			
Geometry	■	■	■	■	■	■	■	■	■	■
Statistics and Probability								■	■	■

Massachusetts Mathematics Curriculum Framework, 2017
Massachusetts Department of Elementary and Secondary Education

Area Model of Multiplication

Area Models for Whole Number Multiplication

3.OA.A.3
Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

2.OA.C.4
Use addition to find the total number of objects arranged in rectangular arrays with up to five rows and up to five columns; write an equation to express the total as a sum of equal addends.

4.NBT.B.5
Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Area Models for Whole Number Multiplication

6×8

$6 \times 8 = 30 + 18 = 48$

8×16

$8 \times 16 = 80 + 48 = 128$

Why does this strategy work?
How does it connect to the properties of operations with real numbers?
How does it connect to the standard algorithm?

Area Models for Whole Number Multiplication

26×17

	10	7
20	200	140
6	60	42

$$
\begin{array}{r}
26 \\
\times\ 17 \\
\hline
42 \\
140 \\
60 \\
+\ 200 \\
\hline
442
\end{array}
$$

Virtual manipulatives

Why does this strategy work?

How does it connect to the properties of operations with real numbers?

How does it connect to the standard algorithm?

How would you work backwards to show division with an area model (5.NBT.B.6)?

Where does it go?

Multiplying Mixed Numbers

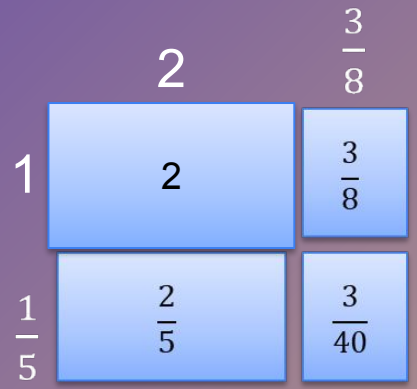
$$2\frac{3}{8} \times 1\frac{1}{5}$$

$$= \left(2 + \frac{3}{8}\right) \times \left(1 + \frac{1}{5}\right)$$

$$= 2 + \frac{3}{8} + \frac{2}{5} + \frac{3}{40}$$

$$= 2 + \frac{34}{40}$$

$$= 2\frac{34}{40}$$



5.NF.B.4
Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

7.NS.A.2
Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide integers and other rational numbers.

Where does it go?

Multiplying Polynomials

Virtual manipulatives

$(2x - 3)(x + 5)$

$= 2x(x) + 2x(5) - 3(x) - 3(5)$

$= 2x^2 + 10x - 3x - 15$

$= 2x^2 + 7x - 15$

	$2x$	-3
x	$2x^2$	$-3x$
$+5$	$10x$	-15

AI.A-APR.A.1a or
MII.A-APR.A.1a
Perform operations on polynomial expressions (addition, subtraction, multiplication), and compare the system of polynomials to the system of integers when performing operations.

Where does it go?

Multiplying Complex Numbers

AII.N-CN.A.2 or MII.N-CN.A.2
Use the relation $i^2 = -1$ and the Commutative, Associative, and Distributive properties to add, subtract, and multiply complex numbers.

$(2 - 3i)(3 + 5i)$

$= 2(3) + 2(5i) - 3i(3) - 3i(5i)$

$= 6 + 10i - 9i - 15i^2$

$= 6 + 10i - 9i + 15$

$= 21 + i$

	2	$-3i$
3	6	$-9i$
$+ 5i$	$10i$	$-15i^2$

Discussion

- How does the idea of an area model for multi-digit whole number multiplication relate to multiplying mixed numbers?
- How does the idea of an area model for multi-digit whole number multiplication relate to multiplying polynomials and complex numbers?
- What other mathematical ideas connect back to the area model for whole number multiplication?

Equivalent Fractions

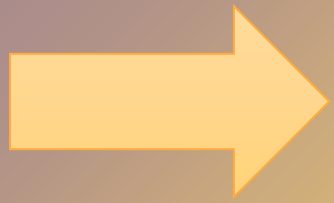
Equivalent Fractions

3.NF.A.3a
Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.

3.NF.A.3b
Recognize and generate simple equivalent fractions, e.g., 1⁄2 = 2⁄4, 4⁄6 = 2⁄3. Explain why the fractions are equivalent, e.g., by using a visual fraction model.

3.NF.A.3c
Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers.

3.NF.A.3d
Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

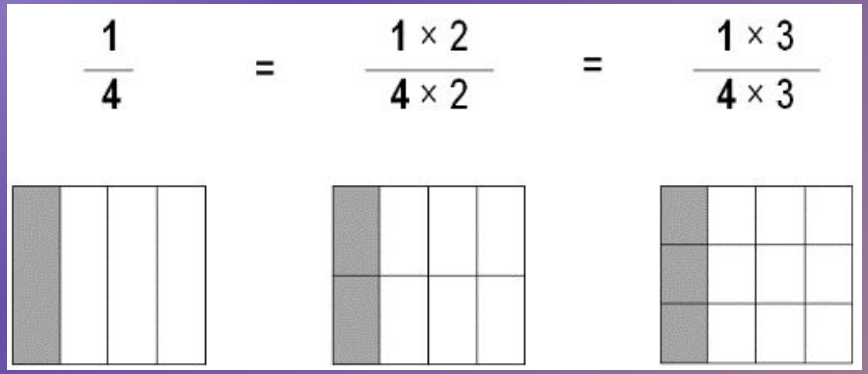


Equivalent Fractions

4.NF.A.1
Explain why a fraction a/b is equivalent to a fraction (n × a)/(n × b) by using visual fraction models, with attention to how the numbers and sizes of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions, including fractions greater than 1.

4.NF.A.2
Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

Equivalent Fractions (Grade 4)

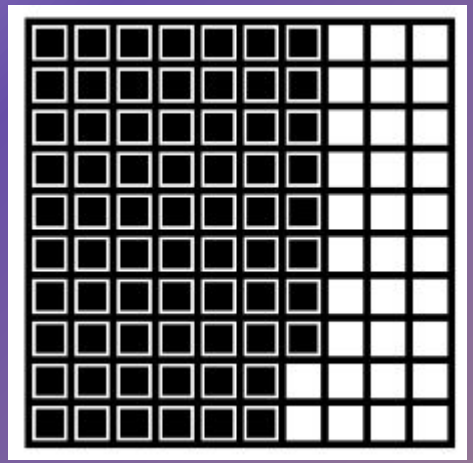


Why does this strategy work?

How does the visual model relate to the numerical method for generating equivalent fractions?

Where does it go?

Equivalent Forms of Rational Numbers



$$\frac{17}{25} = 0.68 = 68\%$$

5.NF.A.1
Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators.

7.EE.B.3
…Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate…

What other models could you use to show equivalent rational numbers?

Where does it go?

Ratios

6.RP.A.3a
Make tables of equivalent ratios relating quantities with whole-number measurements. Find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.

A certain shade of pink paint has 1 part red paint to 3 parts white paint.

Red paint	1 pint	2 pints	3 pints	*n* pints
White paint	3 pints	6 pints	9 pints	3*n* pints

How might you use equivalent fractions to generate a ratio table?

Where does it go?

Percents

6.RP.A.3c
Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.

Quon's smart phone is 36% full with photos. If the smart phone holds 265MB of data, how many MB of data are filled with photos?

$$\frac{36}{100} = \frac{?\ \text{MB}}{256\ \text{MB}}$$

What scale factor could you use to solve this proportion?

How does that approach relate to equivalent fractions?

Where does it go?

Solving Proportions

6.RP.A.3
Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

7.RP.A.3
Use proportional relationships to solve multi-step ratio, rate, and percent problems.

Ibrahim made a necklace that has 1 red bead for every 3 black beads. If he used 35 red beads, how many black beads did he use?

$$\frac{1\ \text{red}}{3\ \text{black}} = \frac{35\ \text{red}}{?\ \text{black}}$$

What scale factor could you use to solve this proportion?

How does that approach relate to equivalent fractions?

Where does it go?

Similar Figures and Scaling

National Math Panel!

7.G.A.1
Solve problems involving scale drawings of geometric figures, such as computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

The scale on a map is that 1 inch on the map represents 25 miles in the real world. How far apart, in the real world, are two towns that are 3 inches apart on the map?

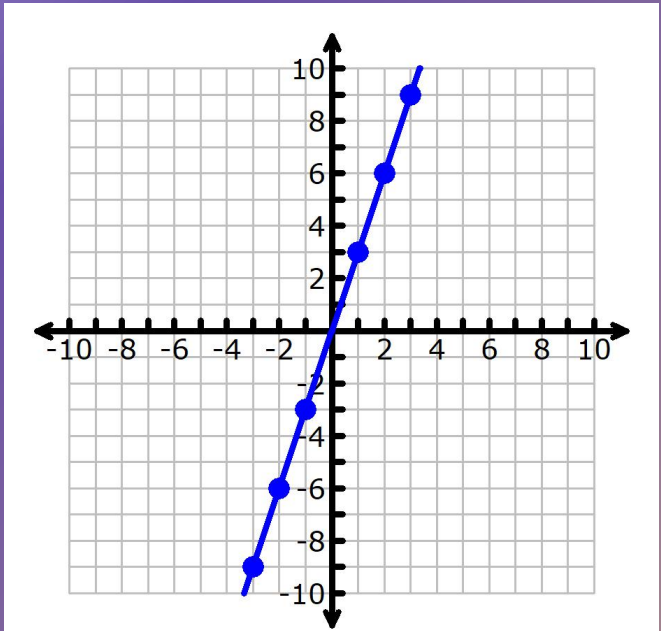
$$\frac{1 \text{ inch}}{25 \text{ miles}} = \frac{3 \text{ inches}}{? \text{ miles}}$$

What scale factor could you use to solve this proportion?

How does that approach relate to equivalent fractions?

Where does it go?

Slope



8.EE.B.6
Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane. Derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.

What scale factor could you use to solve this proportion?

How does that approach relate to equivalent fractions?

Discussion

- How does the idea of equivalent fractions relate to ratio and proportion?
- How does the idea of equivalent fractions relate to similarity?
- How are equivalent fractions, similarity, and slope related?

Number Lines

Number Lines

2.MD.B.6
Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, …, and represent whole-number sums and differences within 100 on a number line diagram.

3.NF.A.2
Understand a fraction as a number on the number line; represent fractions on a number line diagram.

3.NF.A.3a
Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.

4.MD.A.2
…Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

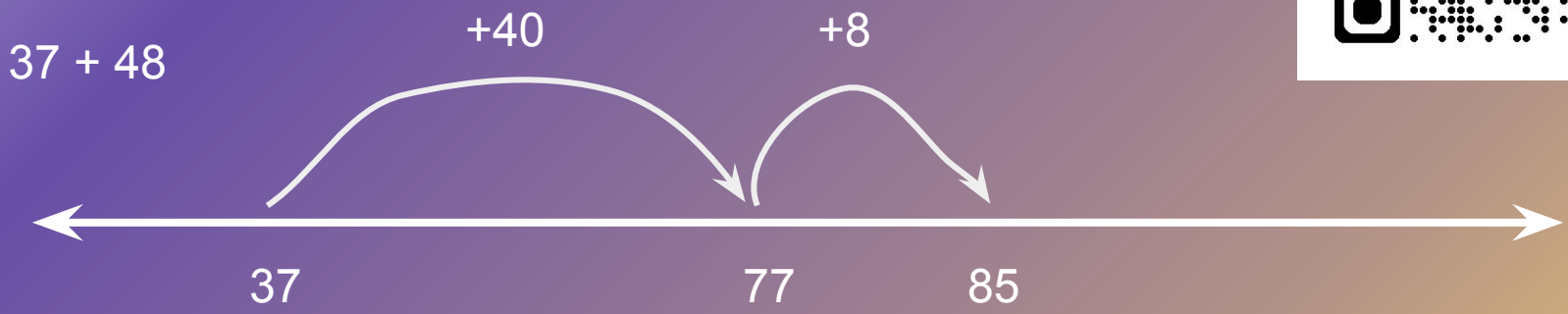
5.G.A.
Determine the corresponding decimal to the tenths or hundredths place of a specified point on a number line.

6.2C
Locate, compare, and order integers and rational numbers using a number line.

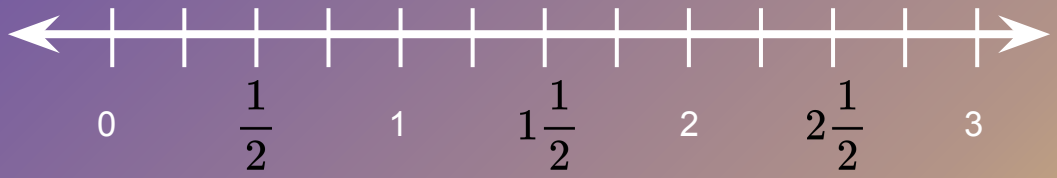
Number Lines

Virtual manipulatives

Open Number Line



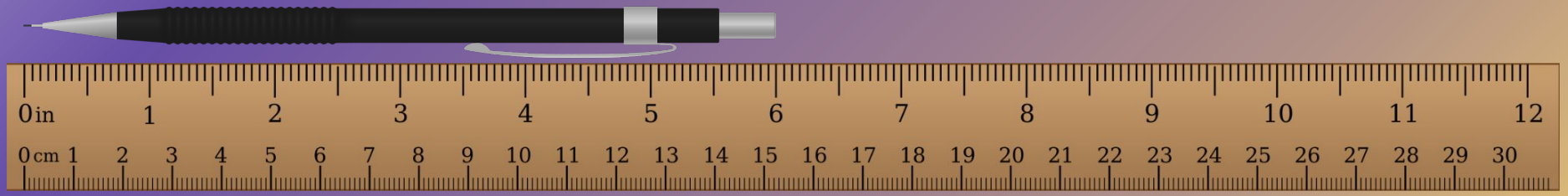
Number Lines



Where does it go?

Linear Measure: Rulers

2.MD.A.1
Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.

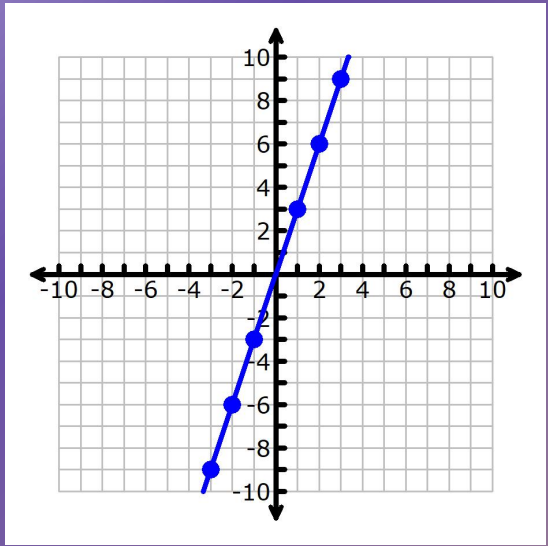


How is a ruler like a number line?

How would the measurement units affect interpreting the number line?

Where does it go?

Cartesian Coordinate System

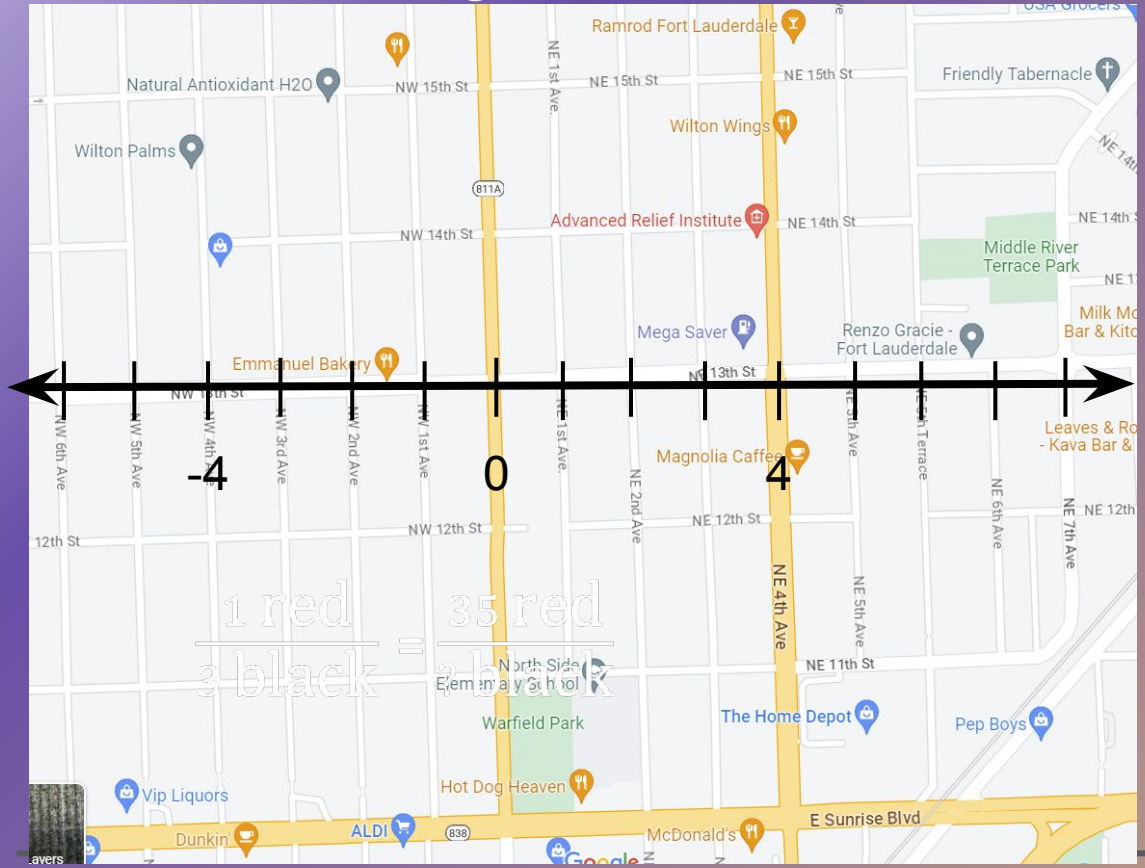


How does a Cartesian coordinate system relate to number lines?

5.G.A.1
Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the zero on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates…

6.NS.8
Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

Where does it go? Absolute Value



6.NS.6a
Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., –(–3) = 3, and that zero is its own opposite.

How does a number line connect with absolute value?

Where does it go?

2.MD.C.7
Tell and write time from analog and digital clocks to the nearest five minutes, using a.m. and p.m.

Telling Time (analog clock)

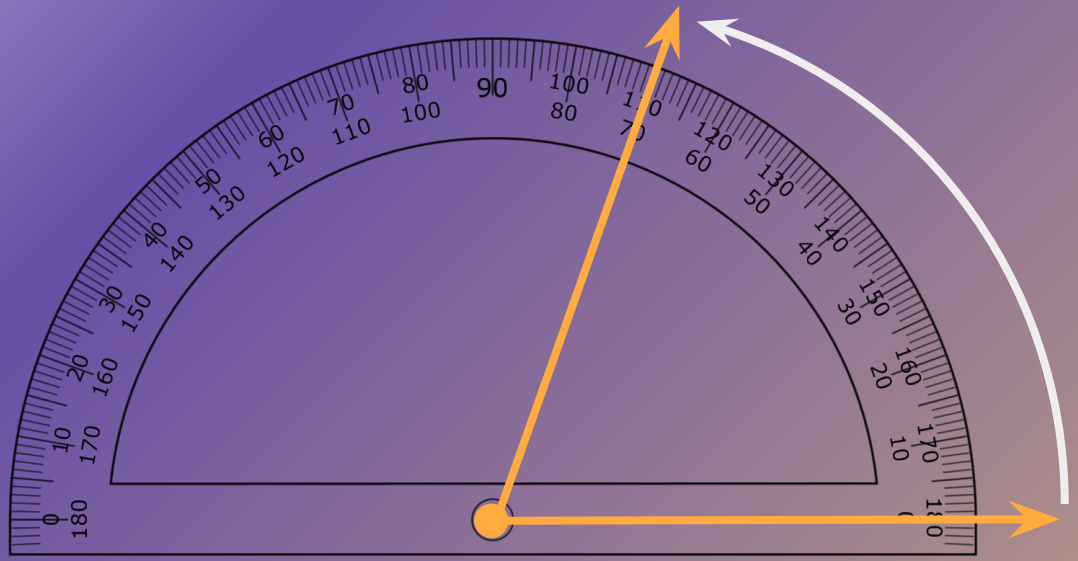
On the curved number line around the circumference of the clock…

What are the units of measure?

What are the major tick marks?

Where does it go?

Angle Measure: Protractors (curved number lines)



4.MD.C.5a
An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1⁄360 of a circle is called a “one-degree angle,” and can be used to measure angles.

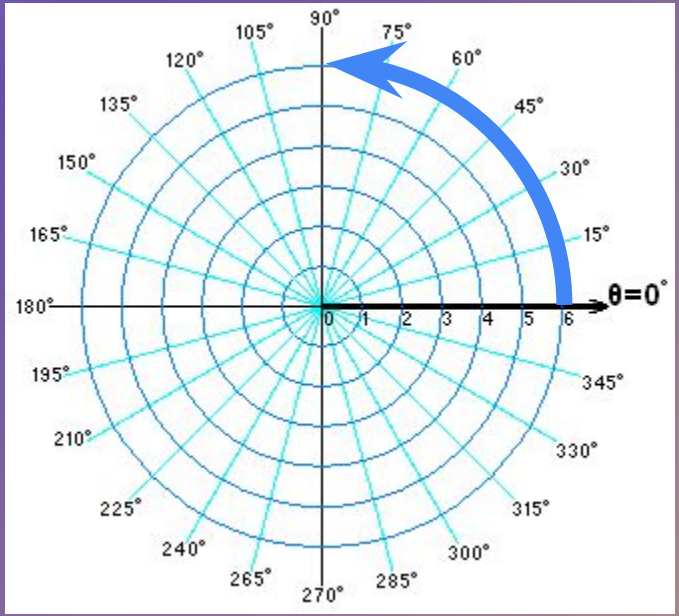
4.MD.C.5b
An angle that turns through *n* one-degree angles is said to have an angle measure of *n* degrees.

4.MD.C.6
Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.

How do the degree marks on the protractor relate to a flat number line?

Where does it go?

Polar Coordinates



P.N-CN-B.4
Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

How do number lines build to polar coordinates?

Questions and Comments

References

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Dr. Paul Gray

NCSM Past President

Cosenza & Associates, LLC

paul@cosenzaassociates.com

www.cosenzaassociates.com

slides

Thank You!