TEKS

AR.2A Determine the patterns that identify the relationship between a function and its common ratio or related finite differences as appropriate, including linear, quadratic, cubic, and exponential functions.

AR.2C Determine the function that models a given table of related values using finite differences and its restricted domain and range.

AR.2D Determine a function that models real-world data and mathematical contexts using finite differences such as the age of a tree and its circumference, figurative numbers, average velocity, and average acceleration.

MATHEMATICAL PROCESS SPOTLIGHT

AR.1F Analyze mathematical relationships to connect and communicate mathematical ideas.

ELPS

5C Spell familiar English words with increasing accuracy, and employ English spelling patterns and rules with increasing accuracy as more English is acquired.

VOCABULARY

finite differences, domain, range, vertex, axis of symmetry, x-intercept, y-intercept, quadratic function

ENGAGE ANSWER:

Possible answer: The number of dots is equal to the term number squared.

1.6 Writing Quadratic Functions

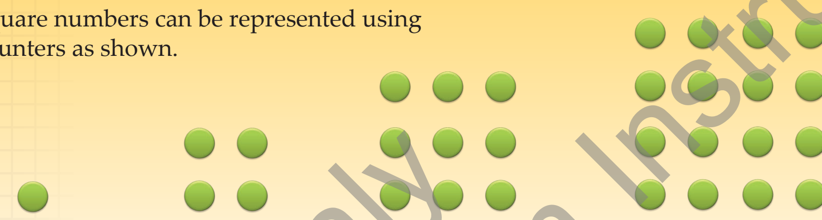
FOCUSING QUESTION What are the characteristics of a quadratic function?

LEARNING OUTCOMES

- I can determine patterns that identify a quadratic function from its related finite differences.
- I can determine the quadratic function from a table using finite differences, including any restrictions on the domain and range.
- I can use finite differences to determine a quadratic function that models a mathematical context.
- I can analyze patterns to connect the table to a function rule and communicate the quadratic pattern as a function rule.

ENGAGE

Square numbers can be represented using counters as shown.

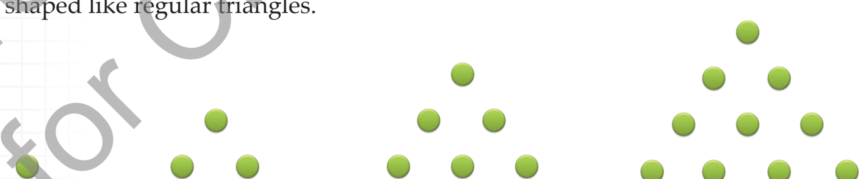


What patterns do you see in the geometric arrangements of square numbers?

See margin.

EXPLORE

A **figurative number**, sometimes called a **figurate number**, is a number that can be represented by a regular geometric arrangement of dots or other objects. For example, **triangular numbers** can be represented using arrangements of dots that are shaped like regular triangles.



2. *In the table, $\Delta x = 1$.*

	Term Number	Triangular Number	
	1	1	
$\Delta x = 2 - 1 = 1$	2	*3*	$\Delta y = 3 - 1 = 2$
$\Delta x = 3 - 2 = 1$	3	*6*	$\Delta y = 6 - 3 = 3$
$\Delta x = 4 - 3 = 1$	4	*10*	$\Delta y = 10 - 6 = 4$
$\Delta x = 5 - 4 = 1$	5	*15*	$\Delta y = 15 - 10 = 5$
$\Delta x = 6 - 5 = 1$	6	*21*	$\Delta y = 21 - 15 = 6$

The data set is not linear because the finite differences in the triangular numbers are not constant.

Term Number	Triangular Number	
1	1	
2	*3*	$\frac{y_n}{y_{n-1}} = \frac{3}{1} = 3$
3	*6*	$\frac{y_n}{y_{n-1}} = \frac{6}{3} = 2$
4	*10*	$\frac{y_n}{y_{n-1}} = \frac{10}{6} = 1\frac{2}{3}$
5	*15*	$\frac{y_n}{y_{n-1}} = \frac{15}{10} = 1\frac{1}{2}$
6	*21*	$\frac{y_n}{y_{n-1}} = \frac{21}{15} = 1\frac{2}{5}$

The data set is not exponential because the successive ratios are not constant.

Use counters to build a sequence of the first six triangular numbers. Record the numbers in a table like the one shown.

TERM NUMBER	TRIANGULAR NUMBER
1	1
2	***3***
3	***6***
4	***10***
5	***15***
6	***21***

1. As you build the sequence, what patterns do you see in each successive term?
See margin.

2. Does the data set follow a linear or an exponential function? Explain your reasoning.
See margin.

3. What patterns do you see in the finite differences or the successive ratios?
See margin.

4. Calculate the second finite difference. What do you notice?
See margin.

5. The quadratic parent function is $y = x^2$. Generate a sequence with y-values for $\{x \mid x = 1, 2, 3, 4, 5, 6\}$.
1, 4, 9, 16, 25, 36

> A set of numbers, such as x-values or y-values, can be represented with braces using set notation. The set of whole numbers less than 10 is represented as {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. If this set is a set of x-values, it can be written as $\{x \mid x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ which is read "the set of all x such that x equals zero, one, two, …"

6. Calculate the second finite differences for the quadratic parent function. What do you notice?

x	y
1	1
2	4
3	9
4	16
5	25
6	36

$\Delta x = 2 - 1 = 1$
$\Delta x = 3 - 2 = 1$
$\Delta x = 4 - 3 = 1$
$\Delta x = 5 - 4 = 1$
$\Delta x = 6 - 5 = 1$

Δy = 4 − 1 = 3
Δy = 9 − 4 = 5
Δy = 16 − 9 = 7
Δy = 25 − 16 = 9
Δy = 36 − 25 = 11

5 − 3 = 2
7 − 5 = 2
9 − 7 = 2
11 − 9 = 2

The second differences are all equal to 2 and are all constant.

1 *For each successive term, you add a row of dots to the bottom of the triangle that is one dot longer than the row of dots at the bottom of the previous triangle.*

2. *See the bottom of page 76.*

3. *Possible answers may include:*

The finite differences are not constant, but they do increase by 1 each time.

The successive ratios can all be represented as $\frac{y_n}{y_{n-1}} = 1 + \frac{2}{n}$.

4. *See below.*

4.

TERM NUMBER	TRIANGULAR NUMBER
1	1
2	***3***
3	***6***
4	***10***
5	***15***
6	***21***

$\Delta x = 2 - 1 = 1$
$\Delta x = 3 - 2 = 1$
$\Delta x = 4 - 3 = 1$
$\Delta x = 5 - 4 = 1$
$\Delta x = 6 - 5 = 1$

Δy = 3 − 1 = 2
Δy = 6 − 3 = 3
Δy = 10 − 6 = 4
Δy = 15 − 10 = 5
Δy = 21 − 15 = 6

3 − 2 = 1
4 − 3 = 1
5 − 4 = 1
6 − 5 = 1

The second differences are all equal to 1 and are all constant.

REFLECT ANSWERS:

The first finite differences in a quadratic function are not constant, but increase or decrease with a particular pattern. The second finite differences are constant in a quadratic function.

The level of finite differences that are constant is the same as the degree of the polynomial (linear: degree one and first differences constant; quadratic: degree two and second differences constant).

ELL STRATEGY

Writing with familiar English language words (ELPS: c5C) helps students both deepen their understanding of the mathematical content as well as become more comfortable with the English language. Having students use the reflect questions for a journal entry in their interactive math notebooks provides an opportunity to reinforce this language proficiency skill.

7. What type of function do you think represents the relationship between the triangular number and the term number, or its position in the sequence?
Possible answer: quadratic function

REFLECT

- **In a linear function, the first finite differences are constant. What is true about the finite differences for a quadratic function?**
  See margin.
- **A linear function contains a polynomial with degree one ($mx + b$) and a quadratic function contains a polynomial with degree two ($ax^2 + bx + c$). What relationship is there between the degree of the polynomial and the level of finite differences that are constant?**
  See margin.

EXPLAIN

In a linear function, the first finite differences, or the difference between consecutive values of the dependent variable, are constant. But for a quadratic function, the first finite differences are not constant. They do, however, have a pattern in that they increase or decrease by the same number. As a result, the second finite differences, or the differences between the first finite differences, are constant.

There are many forms of a quadratic function. **Polynomial form**, also called **standard form**, expresses the function as a polynomial with exponents in decreasing order.

$$f(x) = ax^2 + bx + c$$

In standard form, a, b, and c are rational numbers.

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or click here

Let's look more closely at a quadratic function. The table below shows the relationship between x and $f(x)$ in a quadratic function written in polynomial or standard, $f(x) = ax^2 + bx + c$.

x	PROCESS	$y = f(x)$
0	$a(0)^2 + b(0) + c$	c
1	$a(1)^2 + b(1) + c$	$a + b + c$
2	$a(2)^2 + b(2) + c$	$4a + 2b + c$
3	$a(3)^2 + b(3) + c$	$9a + 3b + c$
4	$a(4)^2 + b(4) + c$	$16a + 4b + c$
5	$a(5)^2 + b(5) + c$	$25a + 5b + c$

$\Delta x = 1 - 0 = 1$ | $\Delta y = (a + b + c) - c = a + b$

$\Delta x = 2 - 1 = 1$ | $\Delta y = (4a + 2b + c) - (a + b + c) = 3a + b$

$\Delta x = 3 - 2 = 1$ | $\Delta y = (9a + 3b + c) - (4a + 2b + c) = 5a + b$

$\Delta x = 4 - 3 = 1$ | $\Delta y = (16a + 4b + c) - (9a + 3b + c) = 7a + b$

$\Delta x = 5 - 4 = 1$ | $\Delta y = (25a + 5b + c) - (16a + 4b + c) = 9a + b$

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The first differences are not constant, but there is a pattern as the differences increase from $a + b$ to $3a + b$, from $3a + b$ to $5a + b$, and so on. So let's look at the second differences. When you look at the second differences, three patterns emerge.

x	$y = f(x)$
0	c
1	$a + b + c$
2	$4a + 2b + c$
3	$9a + 3b + c$
4	$16a + 4b + c$
5	$25a + 5b + c$

$\Delta y = (a + b + c) - c = a + b$

$\Delta y = (4a + 2b + c) - (a + b + c) = 3a + b$

$\Delta y = (9a + 3b + c) - (4a + 2b + c) = 5a + b$

$\Delta y = (16a + 4b + c) - (9a + 3b + c) = 7a + b$

$\Delta y = (25a + 5b + c) - (16a + 4b + c) = 9a + b$

$\Delta^2 y = (3a + b) - (a + b) = 2a$

$\Delta^2 y = (5a + b) - (3a + b) = 2a$

$\Delta^2 y = (7a + b) - (5a + b) = 2a$

$\Delta^2 y = (9a + b) - (7a + b) = 2a$

You can use these three patterns to determine the quadratic function from the table of data.

- The value of c is the y-coordinate of the y-intercept, $(0, c)$.
- The second difference is equal to $2a$.
- The first difference between the y-values for $x = 0$ and $x = 1$ is equal to $a + b$.

FINITE DIFFERENCES AND QUADRATIC FUNCTIONS

In a quadratic function, the second differences between successive y-values are constant if the differences between successive x-values, Δx, are also constant.

If the second differences between consecutive y-values in a table of values are constant, then the values represent a quadratic function.

The formulas for finding the values of a, b, and c to write the quadratic function only work when $\Delta x = 1$. When $\Delta x \neq 1$, there are other formulas that can be used to determine the values of a, b, and c for the quadratic function.

INTEGRATE TECHNOLOGY

Use technology such as a graphing calculator or spreadsheet app on a display screen to show students how, no matter the numbers present in the quadratic function, the second differences will always be constant.

ADDITIONAL EXAMPLES

What type of function (linear, exponential, or quadratic) would best model the data sets below? Justify your answer.

1.

x	1	3	5	7	9
y	15	135	1215	10935	98415

Exponential

2.

x	1	2	3	4	5
y	6	-4	-18	-36	-58

Quadratic

EXAMPLE 1

What type of function would best model the data set below? Justify your answer.

x	y
0	0
1	1
2	6
3	15
4	28

STEP 1 **Determine whether or not the set of data represents a linear function.**

	x	y	
	0	0	
$\Delta x = 1 - 0 = 1$	1	1	$\Delta y = 1 - 0 = 1$
$\Delta x = 2 - 1 = 1$	2	6	$\Delta y = 6 - 1 = 5$
$\Delta x = 3 - 2 = 1$	3	15	$\Delta y = 15 - 6 = 9$
$\Delta x = 4 - 3 = 1$	4	28	$\Delta y = 28 - 15 = 13$

The differences in x, Δx, are all 1, so they are constant.

The differences in y, Δy, are not all the same, so they are not constant.

Therefore, the set of data does not represent a linear function because the first finite differences are not constant.

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STEP 2 Determine whether or not the set of data represents an exponential function.

x	y
0	0
1	1
2	6
3	15
4	28

$\Delta x = 1 - 0 = 1$, $\Delta x = 2 - 1 = 1$, $\Delta x = 3 - 2 = 1$, $\Delta x = 4 - 3 = 1$

$\frac{y_1}{y_0} = \frac{1}{0}$ (undefined), $\frac{y_2}{y_1} = \frac{6}{1} = 6$, $\frac{y_3}{y_2} = \frac{15}{6} = 2.5$, $\frac{y_4}{y_3} = \frac{28}{15} = 1.8666...$

The data set is not exponential because the successive ratios are not constant.

STEP 3 Determine whether or not the set of data represents a quadratic function.

x	y
0	0
1	1
2	6
3	15
4	28

$\Delta x = 1 - 0 = 1$, $\Delta x = 2 - 1 = 1$, $\Delta x = 3 - 2 = 1$, $\Delta x = 4 - 3 = 1$

$\Delta y = 1 - 0 = 1$, $\Delta y = 6 - 1 = 5$, $\Delta y = 15 - 6 = 9$, $\Delta y = 28 - 15 = 13$

$\Delta^2 y = 5 - 1 = 4$, $\Delta^2 y = 9 - 5 = 4$, $\Delta^2 y = 13 - 9 = 4$

The second finite differences are all 4, so the set of data represents a quadratic function.

YOU TRY IT! #1

Determine if the function rule for the set of data is linear, exponential, or quadratic.

x	y
0	0
1	1
2	7
3	19
4	37

Answer: The set of data does not represent any of these functions.

ADDITIONAL EXAMPLES

What type of function (linear, exponential, or quadratic) would best model the data sets below? Justify your answer.

1.

x	-7	-4	-1	2	5
y	2	-16	-34	-52	-70

Linear

2.

x	0	1	2	3	4
y	4	8	16	20	24

None of the functions

INSTRUCTIONAL HINTS

Help students organize their thoughts by creating their own flowchart of steps for determining whether or not a set of data represents a linear, exponential, or quadratic function.

Within the flowchart ask students to include the questions they ask themselves as they look at data such as, "are the differences in x constant?" or "are the successive ratios constant?"

INSTRUCTIONAL HINTS

If students are struggling with "triangular numbers," have them turn back to the Explore on pg. 76 for a visual.

Give students isometric graph paper to draw square or hexagonal numbers. Have them create a table of values and determine the function rule for the set of numbers they drew.

INSTRUCTIONAL HINTS

Encourage students to add abbreviated directions for writing function rules on their flowcharts from the Instructional Hints on pg. 82.

Summarizing learning in one flowchart will help students study and process the material.

ADDITIONAL EXAMPLES

Determine the function rule for the Additional Examples from pages 80 and 81.

pg. 80

1. $y = 5(3)^x$

2. $y = -2x^2 - 4x + 12$

pg. 81

1. $y = -6x - 40$

2. Not linear, exponential, or quadratic

EXAMPLE 2

Determine the function rule for the set of triangular numbers shown below.

x	y
0	0
1	1
2	3
3	6
4	10

STEP 1 **Determine the finite differences between successive x-values and successive y-values.**

Δx	x	y	Δy
	0	0	
$\Delta x = 1 - 0 = 1$			$\Delta y = 1 - 0 = 1$
	1	1	
$\Delta x = 2 - 1 = 1$			$\Delta y = 3 - 1 = 2$
	2	3	
$\Delta x = 3 - 2 = 1$			$\Delta y = 6 - 3 = 3$
	3	6	
$\Delta x = 4 - 3 = 1$			$\Delta y = 10 - 6 = 4$
	4	10	

STEP 2 **Determine whether or not the differences are constant.**

The differences in x, Δx, are all 1, so they are constant.

The differences in y, Δy, are not constant.

STEP 3 **Determine whether or not the second finite differences in successive y-values are constant.**

Δx	x	y	Δy	$\Delta^2 y$
	0	0		
$\Delta x = 1 - 0 = 1$			$\Delta y = 1 - 0 = 1$	
	1	1		$\Delta^2 y = 2 - 1 = 1$
$\Delta x = 2 - 1 = 1$			$\Delta y = 3 - 1 = 2$	
	2	3		$\Delta^2 y = 3 - 2 = 1$
$\Delta x = 3 - 2 = 1$			$\Delta y = 6 - 3 = 3$	
	3	6		$\Delta^2 y = 4 - 3 = 1$
$\Delta x = 4 - 3 = 1$			$\Delta y = 10 - 6 = 4$	
	4	10		

The second differences are all equal to 1 and are constant.

STEP 4 Calculate a, b, and c for the quadratic function $f(x) = ax^2 + bx + c$.
For $x = 0$, $y = 0$. So $c = 0$.

The second finite difference is $2a$, so $2a = 1$ and $a = \frac{1}{2}$.

The first difference between the y-values for $x = 0$ and $x = 1$ is equal to $a + b$, so $a + b = 1$. Since $a = \frac{1}{2}$, b must also equal $\frac{1}{2}$.

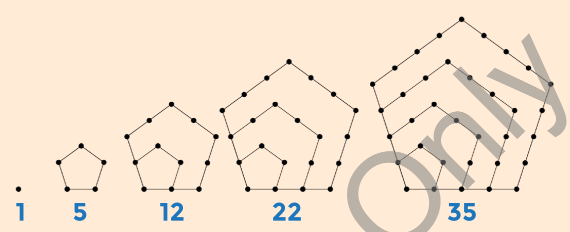
STEP 5 Write the function with the values for a, b, and c:

$f(x) = \frac{1}{2}x^2 + \frac{1}{2}x + 0$ or $f(x) = \frac{x^2 + x}{2}$

YOU TRY IT! #2

Determine the function rule for the set of pentagonal numbers using the values in the table. Use mental math to calculate the finite differences.

x	y
0	0
1	1
2	5
3	12
4	22



Answer: $f(x) = \frac{3}{2}x^2 - \frac{1}{2}x + 0 = \frac{3x^2 - x}{2}$

EXAMPLE 3

Write a quadratic function where the second finite difference is 4, the y-intercept is (0,1), and $a + b$ is 5.

STEP 1 Determine the values of a, b, and c.

The second finite difference is 4. So $2a = 4$, and $a = 2$.
The y-value of the y-intercept, c, is 1.
Since $a + b = 5$ and $a = 2$, then $b = 3$.

ADDITIONAL EXAMPLES

Create a table of values for square numbers from the Engage diagram on pg. 76. Then determine the function rule.

$f(x) = x^2$

Determine the function rule for the set of hexagonal numbers using the values in the table.

x	0	1	2	3	4
y	0	1	6	15	28

$f(x) = 2x^2 - x$

ADDITIONAL EXAMPLES

1. Write a quadratic function where the second finite difference is 3, the y-intercept is (0, -2), and $a + b$ is 13.

$f(x) = 1.5x^2 + 11.5x - 2$

For the data sets shown, write a function rule relating the variables.

2.

x	0	1	2	3	4
y	3	9	17	27	39

$f(x) = x^2 + 5x + 3$

3.

x	1	2	3	4	5
y	16	-4	-28	-56	-88

$f(x) = -2x^2 - 14x + 32$

4.

x	1	2	3	4	5
y	-12	-10	-4	6	20

$f(x) = 2x^2 - 4x - 10$

STEP 2 Write a quadratic function in standard form: $ax^2 + bx + c$.

$f(x) = 2x^2 + 3x + 1$

INSTRUCTIONAL HINT

If students struggle with finding c in YOU TRY IT #3 help them work through a few of the Additional Examples on page 83 before revisiting the YOU TRY IT.

YOU TRY IT! #3

For the data set below, write a function relating the variables.

x	y
1	1
2	9
3	23
4	43
5	69

Answer: $y = 3x^2 - x - 1$

PRACTICE/HOMEWORK

For questions 1 – 8, use finite differences and mental math, as appropriate, to determine if the data sets shown in the tables below represent a linear, exponential, quadratic, or other type of function.

1.

x	$y = f(x)$
1	5
2	11
3	21
4	35
5	53

Quadratic

2.

x	$y = f(x)$
1	5
2	11
3	17
4	23
5	29

Linear

3.

x	$y = f(x)$
1	5
2	9
3	16
4	29
5	52

Exponential

4.

x	$y = f(x)$
1	5
2	14
3	29
4	50
5	77

Quadratic

5.

x	$y = f(x)$
1	5
2	12
3	31
4	68
5	129

Other

6.

x	$y = f(x)$
1	5
2	8
3	13
4	20
5	29

Quadratic

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7.

x	$y = f(x)$
1	5
2	11
3	24
4	53
5	117

Exponential

8.

x	$y = f(x)$
1	5
2	9
3	13
4	17
5	21

Linear

For questions 9 – 12, the data sets shown in the tables represent quadratic functions. Use finite differences to determine the values of a, b, and c and then write the function in standard form.

9.

x	$y = f(x)$
0	7
1	10
2	19
3	34

a = 3, b = 0, c = 7
$f(x) = 3x^2 + 7$

10.

x	$y = f(x)$
0	3
1	6
2	13
3	24

a = 2, b = 1, c = 3
$f(x) = 2x^2 + x + 3$

11.

x	$y = f(x)$
0	-1
1	5
2	19
3	41

a = 4, b = 2, c = -1
$f(x) = 4x^2 + 2x - 1$

12.

x	$y = f(x)$
0	-6
1	-1
2	14
3	39

a = 5, b = 0, c = -6
$f(x) = 5x^2 - 6$

For questions 13 – 16, the data sets shown in the tables represent quadratic functions. Use finite differences to determine f(0), the values of a, b, and c and then write the function in standard form.

13.

x	$y = f(x)$
0	?
1	-1
2	5
3	13
4	23

f(0) = -5; a = 1, b = 3, c = -5
$f(x) = x^2 + 3x - 5$

14.

x	$y = f(x)$
0	?
1	3
2	16
3	41
4	78

f(0) = 2; a = 6, b= -5, c = 2
$f(x) = 6x^2 - 5x + 2$

15.

x	$y = f(x)$
0	?
1	–9
2	–8
3	–1
4	12

$f(0) = -4$; $a = 3$, $b = -8$, $c = -4$
$f(x) = 3x^2 - 8x - 4$

16.

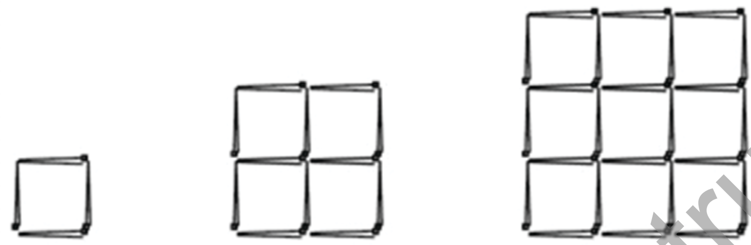
x	$y = f(x)$
0	?
1	7
2	22
3	45
4	76

$f(0) = 0$; $a = 4$, $b = 3$, $c = 0$
$f(x) = 4x^2 + 3x$

For questions 17 – 20 use the situation below.

CRITICAL THINKING

Toothpicks were used to create the pattern below.



1 2 3

17. Relate the length of one side of the figure, x, to the area of the figure, y, by completing the table below. The first row has been completed for you.

LENGTH x	AREA y
1	1
2	*4*
3	*9*

18. Write the function relating the variables in problem 17.
$y = x^2$

19. If the pattern continues, what would be the area of a figure with a side length of 7?
49

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20. Relate the figure number, x, to the total number of toothpicks needed to create the figure, y, by completing the table below. The first row has been completed for you.

FIGURE NUMBER x	TOTAL TOOTHPICKS y
1	4
2	***12***
3	***24***

21. Write the function relating the variables in problem 20.
$y = 2x^2 + 2x$

22. If the pattern continues, how many toothpicks would be needed to create Figure 5?
60

For questions 23 – 24 use the situation below.

SCIENCE

GRAVITY EXPERIMENT

An experiment is conducted by dropping an object from a height of 150 feet and measuring the distance it has fallen at 1-second intervals. Identical objects were used to perform the experiment on Venus, Earth, and Mars. The tables below show the results of each experiment.

VENUS	
TIME (SEC) x	DISTANCE (FEET) y
0	0
1	14.8
2	59.2
3	133.2

EARTH	
TIME (SEC) x	DISTANCE (FEET) y
0	0
1	16
2	64
3	144

MARS	
TIME (SEC) x	DISTANCE (FEET) y
0	0
1	6.2
2	24.8
3	55.8

23. Determine if each table represents a linear, exponential, or quadratic function.
Venus: ***Quadratic*** Earth: ***Quadratic*** Mars: ***Quadratic***

24. Write a function relating the variables in each of the tables above.
Venus: ***$y = 14.8x^2$***
Earth: ***$y = 16x^2$***
Mars: ***$y = 6.2x^2$***