1.7 Modeling with Quadratic Functions

FOCUSING QUESTION How can you use finite differences to construct a quadratic model for a data set?

LEARNING OUTCOMES

- I can use finite differences or common ratios to classify a function as linear, quadratic, or exponential when I am given a table of values.
- I can use finite differences to write a quadratic function that describes a data set.
- I can apply mathematics to problems that I see in everyday life, in society, and in the workplace.

ENGAGE

Mrs. Hernandez wants to construct a rectangular sandbox for her niece and nephew. She has 36 feet of lumber to use as the border. What are some possible dimensions that Mrs. Hernandez could use to construct the sandbox?

EXPLORE

Use color tiles to build rectangles that represent a sandbox with a perimeter of 36. Recall that the area of a rectangle can be found using the formula $A = lw$ and the perimeter of a rectangle can be found using the formula $P = 2(l + w)$. Record the dimensions and the area of each rectangle in a table like the one shown. When you make your table, list the rectangles with widths in order from least to greatest.

WIDTH (IN.)	LENGTH (IN.)	AREA (SQ. IN.)
3	15	45

1. What do you think a scatterplot of area versus width for your set of rectangles would look like?

2. Sketch a scatterplot of area versus width for your set of rectangles.

3. Based on the shape of your graph, what type of function best models the data?

4. Calculate the finite differences between the width and the area. Do the data appear to be linear or quadratic? How do you know?

5. Use the patterns in the finite differences to write a function rule that describes the data set.

6. Use a graphing calculator to graph the function rule over the scatterplot. What do you notice about the function rule and the scatterplot?

7. Compare the domain and range of the data set and the domain and range of the function rule. How are they alike? How are they different?

8. What dimensions will give Mrs. Hernandez a sandbox with the greatest area? Use the table and graph to justify your answer.

9. Are the x-intercepts of the function included in your data set? Why or why not?

10. The perimeter of a rectangle is found using the formula $P = 2l + 2w$. If you know the perimeter is 36 feet, how are the length and width related?

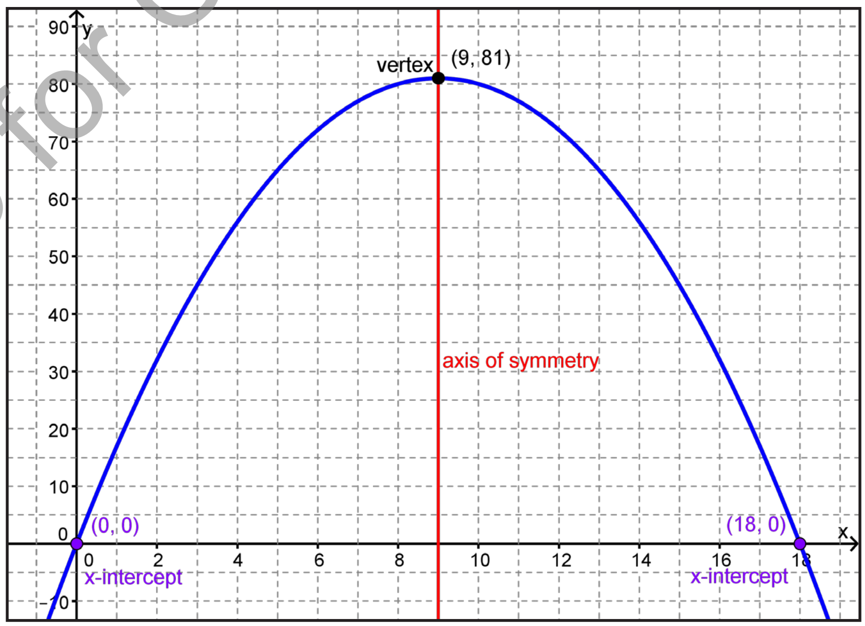
11. Use the relationship between the length and width of a rectangle with a fixed perimeter to write an equation for the area of the rectangle, $A = lw$. How does this equation compare with the function rule that you generated from finite differences?

REFLECT

- **How can you determine a quadratic function model for a data set?**
- **How do the parameters obtained from finite differences relate to the data set being modeled?**

EXPLAIN

Quadratic function models can be used to represent sets of mathematical and real-world data. A quadratic function has several key attributes that are important to consider when using and interpreting models that are based on quadratic functions. The graph of the quadratic function, which is a parabola, helps to explain how these attributes relate to the quadratic function model.



The **vertex of the parabola** represents the data values that generate a minimum or maximum value. In the case of the sandbox problem, the vertex reveals the width, or x-coordinate, that generates the maximum area, or y-coordinate. No other x-value in this model will generate a function value greater than the y-value of the vertex.

Watch Explain and You Try It Videos

or click here

The **axis of symmetry** is a vertical line that represents the dividing line between the part of the parabola with increasing y-values and the part of the parabola with decreasing y-values. The axis of symmetry passes through the vertex.

The **x-intercepts** represent the points with function values that are equal to 0.

In the sandbox problem, you can identify the vertex, axis of symmetry, and x-intercepts from the table of values.

- The **vertex** is the row containing the greatest function value, or the greatest area.
- The **axis of symmetry** is represented by the x-value in the row where the function values change from increasing from row to row to decreasing from row to row.
- The **x-intercept** is a row in which the function value is 0.

		WIDTH (IN.)	LENGTH (IN.)	AREA (SQ. IN.)			
x-INTERCEPT		0	18	0			
	+1				+17		
		1	17	17		-2	
	+1				+15		
		2	16	32		-2	
	+1				+13		
		3	15	45		-2	
	+1				+11		
		4	14	56		-2	
	+1				+9		
		5	13	65		-2	
	+1				+7		
		6	12	72		-2	
	+1				+5		
		7	11	77		-2	
	+1				+3		
		8	10	80		-2	
	+1				+1		
VERTEX		9	9	81		-2	AXIS OF SYMMETRY
	+1				-1		
		10	8	80		-2	
	+1				-3		
		11	7	77		-2	
	+1				-5		
		12	6	72		-2	
	+1				-7		
		13	5	65		-2	
	+1				-9		
		14	4	56		-2	
	+1				-11		
		15	3	45		-2	
	+1				-13		
		16	2	32		-2	
	+1				-15		
		17	1	17		-2	
	+1				-17		
x-INTERCEPT		18	0	0			

You can use the values of the key attributes of a quadratic function in order to interpret the model. In the sandbox problem, the function $f(x) = -x^2 + 18x$ models the data set. The vertex of this function, (9, 81), reveals that a width of 9 feet generates a maximum area of 81 square feet. All x-values to the left of the vertex represent the part of the function where the area increases as the width increases. All x-values to the right of the vertex represent the part of the function where the area decreases as the width increases.

In the sandbox problem, the x-intercepts do not make sense. An x-intercept of (18, 0) means that when the width of the sandbox is 18 feet, the area of the sandbox is 0 square feet. A rectangle cannot have an area of 0 square feet. The x-intercepts represent domain restrictions on the function model when it is applied to the situation.

	DOMAIN	RANGE
FUNCTION	$x \in \mathbb{R}$ (ALL REAL NUMBERS)	$y \leq 81$
SITUATION	$0 < x < 18$ (INCLUDES WHOLE AND REAL NUMBERS)	$0 < y \leq 81$

In the real world, you cannot have lengths that are negative or 0; they must be positive numbers. When using color tiles to represent the situation, you are further limiting the domain to only whole numbers. But in reality, Mrs. Hernandez could create a sandbox with fractional side lengths, such as 3.5 feet by 14.5 feet. Such a sandbox has a perimeter of 36 feet and meets the criteria of the problem.

MODELING WITH QUADRATIC FUNCTIONS

Real-world data rarely follows exact patterns, but you can use patterns in data to look for trends. If the data set has a constant or approximately constant second finite difference, then a quadratic function model may be appropriate for the data set.

The domain and range for the situation may be a subset of the domain and range of the quadratic function model.

EXAMPLE 1

A ball is thrown from the top of a building. The table below shows the height of a ball above the ground at one-second intervals. Determine whether the set of data represents a linear, quadratic, or exponential function.

TIME IN SECONDS, x	HEIGHT IN METERS, $f(x)$
0	100
1	105.1
2	100.4
3	85.9
4	61.6
5	27.5

STEP 1 Determine the finite differences in values of x and values of $f(x)$.

Δx	TIME IN SECONDS, x	HEIGHT IN METERS, $f(x)$	$\Delta f(x)$
	0	100	
$\Delta x = 1 - 0 = 1$			$\Delta f(x) = 105.1 - 100 = 5.1$
	1	105.1	
$\Delta x = 2 - 1 = 1$			$\Delta f(x) = 100.4 - 105.1 = -4.7$
	2	100.4	
$\Delta x = 3 - 2 = 1$			$\Delta f(x) = 85.9 - 100.4 = -14.5$
	3	85.9	
$\Delta x = 4 - 3 = 1$			$\Delta f(x) = 61.6 - 85.9 = -24.3$
	4	61.6	
$\Delta x = 5 - 4 = 1$			$\Delta f(x) = 27.5 - 61.6 = -34.1$
	5	27.5	

STEP 2 Determine the ratios between successive values of $f(x)$.

TIME IN SECONDS, x	HEIGHT IN METERS, $f(x)$
0	100
1	105.1
2	100.4
3	85.9
4	61.6
5	27.5

$\Delta x = 1 - 0 = 1$, $\Delta x = 2 - 1 = 1$, $\Delta x = 3 - 2 = 1$, $\Delta x = 4 - 3 = 1$, $\Delta x = 5 - 4 = 1$

$\frac{y_n}{y_{n-1}} = \frac{105.1}{100} \approx 1.051$

$\frac{y_n}{y_{n-1}} = \frac{100.4}{105.1} \approx 0.955$

$\frac{y_n}{y_{n-1}} = \frac{85.9}{100.4} \approx 0.856$

$\frac{y_n}{y_{n-1}} = \frac{61.6}{85.9} \approx 0.717$

$\frac{y_n}{y_{n-1}} = \frac{27.5}{61.6} \approx 0.446$

STEP 3 Determine the second finite differences in the $f(x)$ values.

TIME IN SECONDS, x	HEIGHT IN METERS, $f(x)$
0	100
1	105.1
2	100.4
3	85.9
4	61.6
5	27.5

$\Delta x = 1 - 0 = 1$, $\Delta x = 2 - 1 = 1$, $\Delta x = 3 - 2 = 1$, $\Delta x = 4 - 3 = 1$, $\Delta x = 5 - 4 = 1$

First differences: +5.1, -4.7, -14.5, -24.3, -34.1

Second differences: -9.8, -9.8, -9.8, -9.8

STEP 4 Determine whether the finite differences or the ratios between successive values of $f(x)$ are approximately constant.

- The first finite differences range in value from 5.1 to –34.1. This is a wide range, so the first finite differences are not approximately constant.
- The ratios between successive values of $f(x)$ range from 0.446 to 1.051. This is also a wide range, so the ratios between successive values of $f(x)$ are not approximately constant.
- The second finite differences are all –9.8 and are constant.

The set of data represents a quadratic function, rather than a linear or exponential function, because the differences in x are constant and the second finite differences in f(x) are constant.

YOU TRY IT! #1

A softball pitcher throws a ball to her catcher. The ball's path is tracked in the table.

HORIZONTAL DISTANCE FROM PITCHER IN FEET, x	VERTICAL HEIGHT IN FEET, $f(x)$
0	2
5	5.6
10	8.4
15	10.4
20	11.6
25	12

Determine whether the relationship is linear, exponential, or quadratic.

EXAMPLE 2

The total amount in a savings account is shown in the table. Determine whether the interest that is being earned in the savings account follows a linear, quadratic, or exponential function.

1-YEAR INTERVAL, x	INTEREST IN DOLLARS, $f(x)$
0	500
1	530
2	561.80
3	595.51
4	631.24

STEP 1 Determine the finite differences in values of x and values of $f(x)$.

1-YEAR INTERVAL, x	INTEREST IN DOLLARS, $f(x)$
0	500
1	530
2	561.80
3	595.51
4	631.24

$\Delta f(x) = 530 - 500 = 30$

$\Delta f(x) = 561.80 - 530 = 31.80$

$\Delta f(x) = 595.51 - 561.80 = 33.71$

$\Delta f(x) = 631.24 - 595.51 = 35.73$

The differences in the x-values are constant and the first finite differences in the values for $f(x)$ range from 30 to 35.73.

STEP 2 Determine the ratios between successive values of $f(x)$.

1-YEAR INTERVAL, x	INTEREST IN DOLLARS, $f(x)$
0	500
1	530
2	561.80
3	595.51
4	631.24

$\frac{y_n}{y_{n-1}} = \frac{530}{500} \approx 1.06$

$\frac{y_n}{y_{n-1}} = \frac{561.80}{530} \approx 1.06$

$\frac{y_n}{y_{n-1}} = \frac{595.51}{561.80} \approx 1.061$

$\frac{y_n}{y_{n-1}} = \frac{631.24}{595.51} \approx 1.059$

The ratios between successive values of f(x) are approximately 1.06. These values are all close together, so the ratios between successive values of f(x) are approximately constant, and the data set represents an exponential function.

YOU TRY IT! #2

The total amount in a savings account is shown in the table. Determine whether the interest that is being earned in the savings account follows a linear, quadratic, or exponential function.

1-YEAR INTERVAL, x	TOTAL AMOUNT IN DOLLARS, $f(x)$
0	750
1	780
2	810
3	840
4	870

EXAMPLE 3

Possum Kingdom Lake in Palo Pinto County, Texas, was the setting for a world-class cliff diving in 2014. The champion diver's approximate position during the dive is recorded in the table.

DISTANCE AWAY FROM THE CLIFF IN METERS, x	HEIGHT ABOVE THE WATER IN METERS, $f(x)$
0	27
1	28.1
2	27.4
3	24.8
4	20.0
5	12.9

Data Source: Redbullcliffdiving.com

Use the data set to generate a quadratic function that best models the data.

Use the table to estimate the height of the cliff, the height of the diver at his highest point, and his distance from the cliff when he entered the water.

STEP 1 **Determine the finite differences in x-values and the second finite differences in the values of $f(x)$.**

	DISTANCE AWAY FROM THE CLIFF IN METERS, x	HEIGHT ABOVE THE WATER IN METERS, $f(x)$		
	0	27		
$\Delta x = 1 - 0 = 1$			+1.1	
	1	28.1		-1.8
$\Delta x = 2 - 1 = 1$			-0.7	
	2	27.4		-1.9
$\Delta x = 3 - 2 = 1$			-2.6	
	3	24.8		-2.2
$\Delta x = 4 - 3 = 1$			-4.8	
	4	20.0		-2.3
$\Delta x = 5 - 4 = 1$			-7.1	
	5	12.9		

STEP 2 **Calculate the average of the second finite differences and use this value to determine a in your quadratic function model, $f(x) = ax^2 + bx + c$.**

$$2a = \frac{-1.8 - 1.9 - 2.2 - 2.3}{4} = -2.05$$

So $a = -1.025$

STEP 3 **Calculate the value of b.**

The difference between the values of $f(x)$ for $x = 0$ and 1 is $(a + b)$.

$$a + b = 1.1$$
$$(-1.025) + b = 1.1$$
$$b = 2.125$$

STEP 4 Determine the value of c.

The value of $f(0) = c$. $f(0) = 27$, so $c = 27$.

STEP 5 Substitute the values of a, b, and c into the general form to determine the function model.

The quadratic function model is $f(x) = -1.025x^2 + 2.125x + 27$.

Using the table, the top of the cliff must have been about 27 meters above the water. The highest point of the dive was a little more than 28 meters, and the diver entered the water at a distance of a little more than 6 meters from the cliff.

YOU TRY IT! #3

A study compared the speed x (in miles per hour) and the average fuel economy $f(x)$ (in miles per gallon) for cars. The results in 10 mile per hour increments over 20 mph are shown in the table.

10-MILE PER HOUR INTERVAL, x	MILES PER HOUR	GASOLINE USAGE IN MILES PER GALLON, $f(x)$
0	20	24.5
1	30	28.0
2	40	30.0
3	50	30.2
4	60	28.8
5	70	25.8

Use the data set to generate a quadratic model. Use your model to predict the fuel economy at 80 miles per hour.

PRACTICE/HOMEWORK

For questions 1-6 determine whether the set of data represents a linear, quadratic, or exponential function.

1.

x	$y = f(x)$
1	7
2	16
3	27
4	40
5	55

2.

x	$y = f(x)$
1	-4
2	-1
3	2
4	5
5	8

3.

x	$y = f(x)$
1	-13
2	-28
3	-45
4	-64
5	-85

4.

x	$y = f(x)$
1	2
2	4
3	8
4	16
5	32

5.

x	$y = f(x)$
1	-4
2	-6
3	-6
4	-4
5	0

6.

x	$y = f(x)$
1	0.2
2	0.04
3	0.008
4	0.0016
5	0.00032

For questions 7-12 use the data set to generate a quadratic function that best models the data.

7.

x	$y = f(x)$
1	3
2	12
3	27
4	48
5	75

8.

x	$y = f(x)$
1	2
2	2
3	0
4	-4
5	-10

9.

x	$y = f(x)$
1	-12
2	-20
3	-24
4	-24
5	-20

10.

x	$y = f(x)$
1	8.5
2	18
3	28.5
4	40
5	52.5

11.

x	$y = f(x)$
1	1
2	-8
3	-23
4	-44
5	-71

12.

x	$y = f(x)$
1	6
2	28
3	58
4	96
5	142

For questions 13 and 14, use the following information.

SCIENCE

The Texas Department of Public Safety can use the length of skid marks to help determine the speed of a vehicle before the brakes were applied. The quadratic function that best models the data is $f(x) = \frac{x^2}{24}$ where x represents the speed of the vehicle and $f(x)$ is the length of the skid mark. The speeds of a vehicle and the length of the corresponding skid marks are shown in the table below.

SPEED OF A VEHICLE IN MILES PER HOURS, x	DISTANCE OF THE SKID IN FEET, $f(x)$
30	37.5
36	54
42	73.5
48	96
54	121.5
60	150

13. Use the table of data to determine the length of a skid mark of a vehicle that was traveling at a speed of 72 miles when it applied the brakes.

14. Use the table of data to determine how fast a vehicle was traveling if the length of the skid mark was 24 feet.

For questions 15 - 17, use the following information.

SCIENCE

A ball is thrown upward with an initial velocity of 35 meters per second. The position of the ball over time is recorded in the table below.

15. Use the data in the table to generate a quadratic function that models the data.

16. Use the data in the table to find the height of the ball after 7 seconds.

17. Use the data in the table to determine after how many seconds the ball will be 30 meters high.

TIME IN SECONDS, x	DISTANCE FROM THE GROUND IN METERS, $f(x)$
0	0
1	30
2	50
3	60
4	60
5	50

For questions 18 - 20, use the following information.

GEOMETRY

Judy wants to construct a rectangular pen for her puppy, but only has 56 feet of fencing to use for the pen. The table below shows the width, length, and area of different size pens.

WIDTH (FT)	LENGTH (FT)	AREA (SQ. FT.)
10	18	180
11	17	187
12	16	192
13	15	195
14	14	196
15	13	195
16	12	192

18. Use the data in the table to generate a quadratic function that models the data.

19. Use the data in the table to determine the dimensions that would create a pen with an area of 160 ft^2.

20. Use the data in the table to determine the area of a pen where one of the dimensions measures 20 feet.