Algebraic Reasoning

A New High School Course in Texas

Participant Handout Packet

September 21, 2016

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Workshop facilitated at Region 10 ESC

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Table of Contents

- Blank graph paper
- Lesson 1 – Inverses of Functions
- Lesson 2 – Factoring Polynomials with Graphs and Tables
- Lesson 3 – Solving Systems of Three Linear Equations
- Case Study: Analyzing Course Pathways

Lessons are from Gray, Weilmuenster, & Hylemon. (2016). *Algebraic Reasoning*. Cosenza & Associates, LLC: Dallas, TX.

3.1 Generating Inverses of Functions

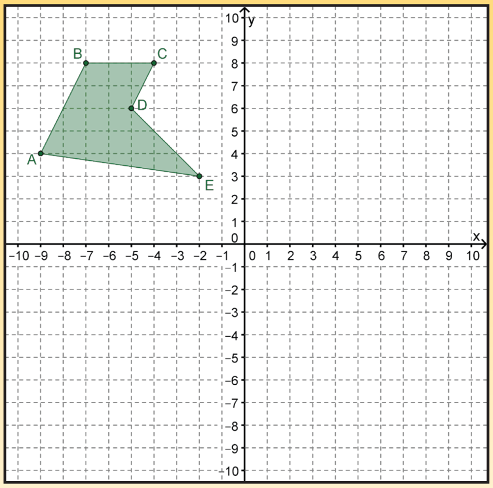
FOCUSING QUESTION What is the inverse of a function?

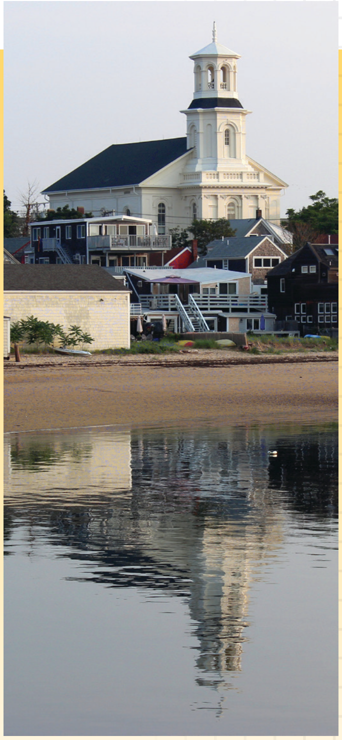
LEARNING OUTCOMES

- I can compare and contrast the key attributes of a function and its inverse when I have the function as a table, graph, or written symbolically.
- I can represent the domain and range of a linear function in a variety of ways, including interval notation, inequalities, and set builder notation.
- I can use and select tools, including graphing technology, paper and pencil, and manipulatives like patty paper, to solve problems.

ENGAGE

Dylan, a computer animator, needs to reflect the figure shown as a part of an animation process.





Provincetown Library
Provincetown, Massachusetts

What would be the coordinates of each vertex in the new figure if Dylan reflected the original figure across the y-axis? The x-axis?

EXPLORE

The distance required to stop a moving vehicle is a function of the speed of the vehicle. According to the Texas Driver Handbook, the distance required to stop a vehicle moving at a given speed, on dry pavement with good tires, is shown in the table on page 264.

SPEED (MILES PER HOUR)	20	30	40	50	60	70
BRAKING DISTANCE (FEET)	63	109	164	229	303	387

Source: Texas Department of Public Safety

1. Use finite differences or successive ratios to determine if the data set represents a linear, quadratic, cubic, or exponential function.

2. Make a scatterplot of the braking distance versus speed.

3. Suppose that you knew from a skid mark the length of the braking distance and wanted to know the speed of the moving vehicle. In this situation, what would be the independent variable and what would be the dependent variable?

When describing a scatterplot, you can say the scatterplot is the name of the **dependent variable** versus the name of the **independent variable**. For example, if you are graphing data where **speed is the independent variable** and **distance is the dependent variable**, you can say that you have a scatterplot of ***distance*** versus ***speed***.

4. On the same grid, make a scatterplot of speed versus braking distance. What do you notice about the two scatterplots?

5. Draw the line $y = x$ on your graph. Lay a sheet of patty paper on top of the graph. Trace and label the x-axis, y-axis, the line $y = x$, and the points in the first scatter plot, braking distance versus speed.

6. Use the patty paper to reflect the first scatterplot across the line $y = x$. *(Hint: Hold the patty paper along the top-right corner of the line $y = x$ with your right index finger and thumb. Hold the patty paper along the bottom-left corner of the line $y = x$ with your left index finger and thumb. Flip the patty paper over without moving your fingers and thumbs. Line up the axes on the patty paper with the axes on the graph beneath.)* What do you notice?

The **inverse** of a function is a relation in which the domain and range of the original function are switched. The domain of the original function becomes the range of the inverse and the range of the original function becomes the domain of the inverse. The scatterplot of speed versus braking distance is the inverse of the scatterplot of braking distance versus speed.

7. How are the inverse scatterplot and the original scatterplot related?

8. A table of values for both scatterplots is shown.

Braking Distance vs. Speed

SPEED, x (MILES PER HOUR)	BRAKING DISTANCE, y (FEET)
20	63
30	109
40	164
50	229
60	303
70	387

Speed vs. Braking Distance

BRAKING DISTANCE, x (FEET)	SPEED, y (MILES PER HOUR)
63	20
109	30
164	40
229	50
303	60
387	70

How are the domain and range of the original scatterplot (braking distance versus speed) and the inverse scatterplot (speed versus braking distance) related?

9. Graph the function $f(x) = 3x - 4$. Use a graph and a table of values to represent the inverse of $f(x)$. How do the slope and intercepts of the function compare with the slope and intercepts of the inverse?

10. Graph the function $g(x) = (x + 4)^2 + 2$. Use a graph and a table of values to represent the inverse of $g(x)$. How does the vertex of $g(x)$ compare to the vertex of its inverse?

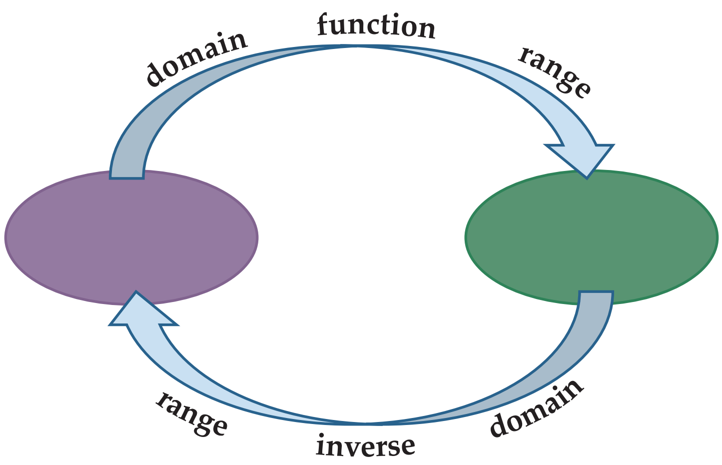
REFLECT

- **How can you use a graph to generate the inverse of a function?**
- **How can you use a table to generate the inverse of a function?**
- **How are the domain and range of a function related to the domain and range of its inverse?**
- **Is the inverse of a function always a new function? Explain your answer.**

EXPLAIN

A function is a relationship between an independent variable and a dependent variable. The values of the independent variable are called the domain of the function and the values of the dependent variable are called the range of the function.

But what happens if the relationship is reversed and the range values become the input while the domain values become the output? That situation is called an **inverse** relation. The range of the original function becomes the domain of the inverse relation, and the domain of the original function becomes the range of the inverse relation.



You can generate inverses of functions using tables, graphs, or equations.

INVERSES IN TABLES

Fahrenheit and Celsius are two different units that are used to measure temperature. The tables below show some ordered pairs that represent equivalent temperatures in each scale. The left-hand table assumes that you know the temperature in degrees Celsius (i.e., Celsius temperature is the independent variable) and you want to determine the temperature in degrees Fahrenheit (i.e., Fahrenheit temperature is the dependent variable). The right-hand table assumes that you know the temperature in degrees Fahrenheit (i.e., Fahrenheit temperature is the independent variable) and you want to determine the temperature in degrees Celsius (i.e., Celsius temperature is the dependent variable).

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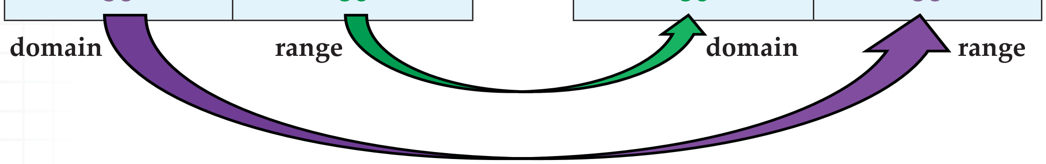
Celsius to Fahrenheit, $F(x)$

TEMPERATURE IN CELSIUS	TEMPERATURE IN FAHRENHEIT
$-17\frac{7}{9}$	0
-10	14
0	32
10	50
20	68
30	86

Fahrenheit to Celsius, $C(x)$

TEMPERATURE IN FAHRENHEIT	TEMPERATURE IN CELSIUS
0	$-17\frac{7}{9}$
14	-10
32	0
50	10
68	20
86	30

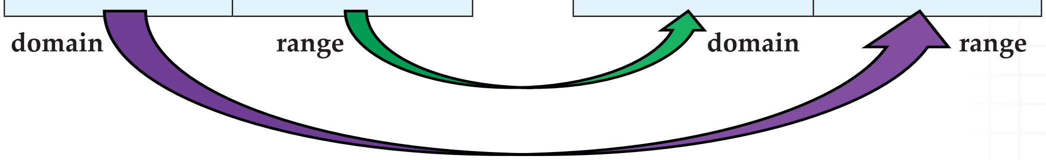


Celsius to Fahrenheit, $F(x)$

TEMPERATURE IN CELSIUS	TEMPERATURE IN FAHRENHEIT
$-17\frac{7}{9}$	0
-10	14
0	32
10	50
20	68
30	86

Fahrenheit to Celsius, $C(x)$

TEMPERATURE IN FAHRENHEIT	TEMPERATURE IN CELSIUS
0	$-17\frac{7}{9}$
14	-10
32	0
50	10
68	20
86	30



The domain of $F(x)$, the Celsius to Fahrenheit conversion function, becomes the range of $C(x)$, the Fahrenheit to Celsius conversion. The range of $F(x)$, the Celsius to Fahrenheit conversion function, becomes the domain of $C(x)$, the Fahrenheit to Celsius conversion. Thus, $C(x)$ is the inverse of $F(x)$.

Also notice that the y-intercept of $F(x)$, (0, 32), becomes the x-intercept of $C(x)$, (32, 0), since the x-values and y-values switch. Likewise, the x-intercept of $F(x)$, $(-17\frac{7}{9}, 0)$, becomes the y-intercept of $C(x)$, $\left(0, -17\frac{7}{9}\right)$.

ELPS CONNECTION

As you read the next section, use the graph as visual support and what you know about temperature as contextual support to identify and develop key vocabulary terms.

INVERSES IN GRAPHS

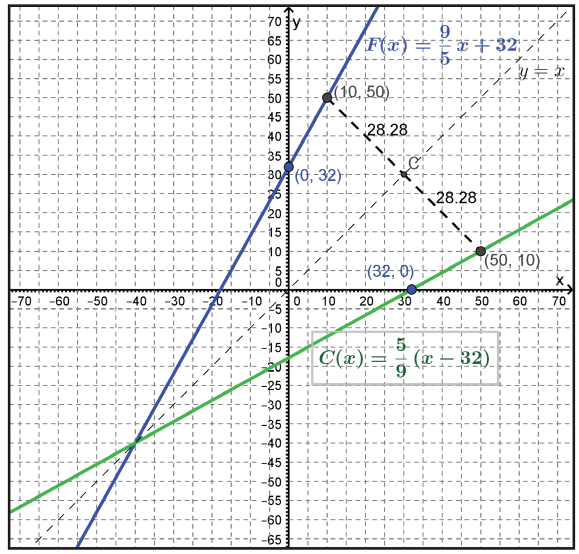
In a graph, an inverse of a function appears as a reflection of the graph of the original function across the line $y = x$. This reflection transforms the ordered pairs, (x, y) of the original function into the ordered pairs (y, x) for the inverse function. The x- and y-coordinates of the original function are switched to generate the inverse.

The Fahrenheit-Celsius conversion formulas, where F represents the temperature in degrees Fahrenheit and C represents the temperature in degrees Celsius are shown.

- $F = \frac{9}{5}C + 32$
- $C = \frac{5}{9}(F - 32)$

For each function, let the independent variable be x. Then, $F(x)$ will give you the temperature in degrees Fahrenheit if you know the temperature in Celsius, x. Likewise, $C(x)$ will give you the temperature in degrees Celsius if you know the temperature in degrees Fahrenheit, x.

Notice that each line is a reflection of the other across the line $y = x$. Graphically, reflections have the property that each point on the graph of the function is the same distance from the line of reflection as its image, or reflection, on the graph of the inverse. For example, (10, 50) represents the equivalent temperatures 10°C and 50°F. This point lies 28.28 units from the line $y = x$, which is the line of reflection. Its image, (50, 10), represents the equivalent temperatures 50°F and 10°C. This point also lies 28.28 units from the line $y = x$. Thus, the points are equidistant, or the same distance, from the line of reflection.



Also notice the relationships between the intercepts.

- The y-intercept of $F(x)$ becomes the x-intercept of $C(x)$. In other words, the point (0, 32) on the graph of the original function becomes the point (32, 0) on the graph of the inverse because the x- and y-coordinates for the inverse are switched from the original function.
- The x-intercept of $F(x)$ becomes the y-intercept of $C(x)$. In other words, the point $\left(-17\frac{7}{9}, 0\right)$ on the graph of the original function becomes the point $\left(0, -17\frac{7}{9}\right)$ on the graph of the inverse because the x- and y-coordinates for the inverse are switched from the original function.

INVERSES IN EQUATIONS

The Fahrenheit-Celsius conversion formulas, where F represents the temperature in degrees Fahrenheit and C represents the temperature in degrees Celsius are shown.

- $F = \frac{9}{5}C + 32$
- $C = \frac{5}{9}(F - 32)$

You can determine the inverse of a function from its equation. Because the domain and range of the original function are switched to generate the inverse, switch the variables that represent the independent and dependent variables in the equation and solve for the dependent variable.

Begin with the Celsius to Fahrenheit conversion formula written as a function, $F(x) = \frac{9}{5}x + 32$. x represents the independent variable and $F(x)$ represents the dependent variable, which can also be rewritten as y. For its inverse, switch x and y and then solve for y.

$$y = \frac{9}{5}x + 32$$

$$x = \frac{9}{5}y + 32$$

$$x - 32 = \frac{9}{5}y + 32 - 32$$

$$x - 32 = \frac{9}{5}y$$

$$\frac{5}{9}(x - 32) = \frac{5}{9}\left(\frac{9}{5}\right)y$$

$$\frac{5}{9}(x - 32) = y$$

In this inverse equation, y represents $C(x)$, which is the temperature in degrees Celsius, and x represents the temperature in degrees Fahrenheit. This equation, which is a linear function, is equivalent to the Fahrenheit to Celsius temperature conversion formula.

KEY ATTRIBUTES OF INVERSES OF FUNCTIONS

The inverse of a function may or may not be a function itself. However, it does have several important key attributes:

- The domain of the original function is the range of the inverse.
- The range of the original function is the domain of the inverse.
- If the inverse of $f(x)$ is a function, it can be written with the notation $f^{-1}(x)$, which is read, "f inverse of x."
- The x-intercept(s) of the original function is/are the y-intercept(s) of the inverse.
- The y-intercept of the original function is the x-intercept of the inverse.
- The maximum or minimum function value (y-value) of the original function (e.g., vertex of a quadratic function) is the maximum or minimum x-value of the inverse.

EXAMPLE 1

Generate the inverse of the exponential function represented in the table that follows. Is the inverse a function? Justify your answer.

x	y
– 2	$\frac{1}{9}$
– 1	$\frac{1}{3}$
0	1
1	3
2	9

STEP 1 Determine the inverse by switching the domain and range values of the original function.

x	y
$\frac{1}{9}$	– 2
$\frac{1}{3}$	– 1
1	0
3	1
9	2

STEP 2 Evaluate the domain and range of the inverse to determine whether or not the inverse is a function.

Each domain value in the inverse's table results in a single range value in the inverse's table. The relationship between the original exponential function's domain and range is related to the relationship of the domain and range of the inverse. Therefore, it is reasonable to conclude that the inverse is also a function.

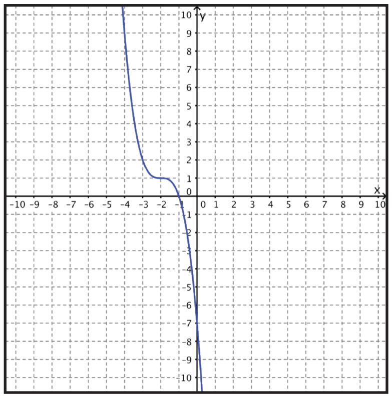
YOU TRY IT! #1

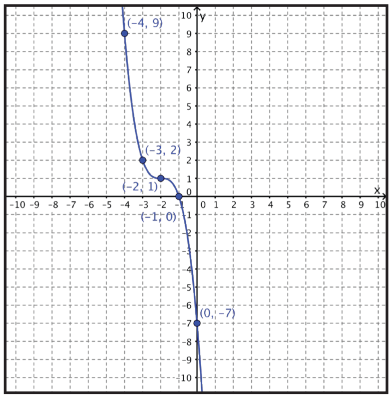
Generate the inverse of the absolute value function represented in the table to the right. If the inverse is a function, write the equation of the inverse function. If the inverse is not a function, explain why not.

x	y
– 1	3
0	0
1	– 3
2	0
3	3

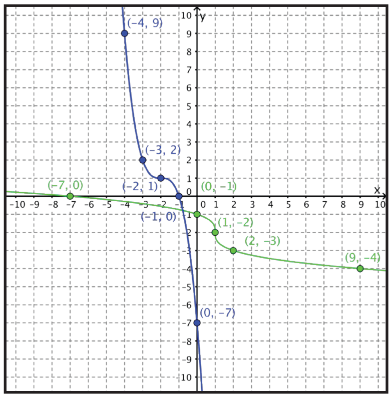
EXAMPLE 2

Generate the graph of the inverse of the cubic function $f(x) = -(x + 2)^3 + 1$ whose graph is shown below.

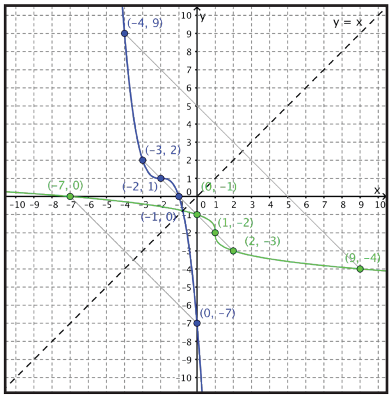




STEP 1 Determine several points on the original graph that represent the general shape of the curve as well as key attributes of the function, such as intercepts.



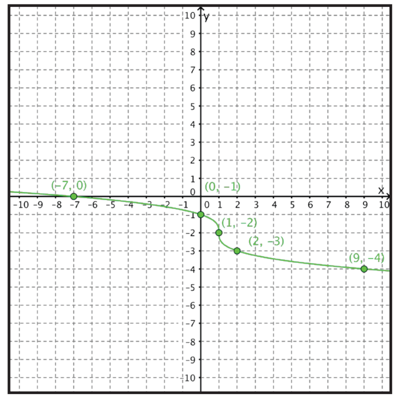
STEP 2 Switch the x- and y-values of each of the chosen points from the original graph and graph them on the coordinate plane. Then draw the curve of the inverse through the points you graphed on the coordinate plane.



STEP 3 Verify visually that the graphs are reflections of one another across the line $y = x$ and that the graphed points are the same distance from the line $y = x$.

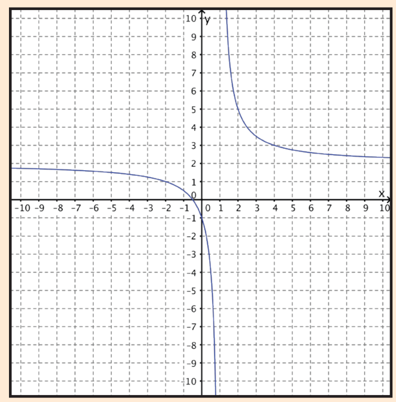
Notice that the y-intercept of the original function is the x-intercept of the inverse and the x-intercept(s) of the original function are the y-intercept(s) of the inverse.

The graph of the inverse of $f^{-1}(x)$ is shown.



YOU TRY IT! #2

Generate the graph of the inverse of the rational function $g(x) = \frac{3}{x-1} + 2$ whose graph is shown below.



EXAMPLE 3

Generate the equation of $h^{-1}(x)$, the inverse of the quadratic function $h(x) = 3(2x + 1)^2 - 7$.

STEP 1 Rewrite the function using x as the independent variable and y as the dependent variable.

$$y = 3(2x + 1)^2 - 7$$

STEP 2 Reverse the variables x and y in the equation and solve the equation for y.

$$x = 3(2y+1)^2 - 7$$

$$x + 7 = 3(2y+1)^2 - 7 + 7$$

$$\frac{x+7}{3} = \frac{3(2y+1)^2}{3}$$

$$\pm\sqrt{\frac{x+7}{3}} = \sqrt{(2y+1)^2}$$

$$-1 \pm \sqrt{\frac{x+7}{3}} = 2y + 1 - 1$$

$$\frac{-1 \pm \sqrt{\frac{x+7}{3}}}{2} = \frac{2y}{2}$$

$$\frac{-1 \pm \sqrt{\frac{x+7}{3}}}{2} = y$$

STEP 3 **Rewrite the inverse as an equation.**

inverse of $h(x)$ is $y = \dfrac{-1 \pm \sqrt{\frac{x+7}{3}}}{2}$

YOU TRY IT! #3

Generate the equation of $p^{-1}(x)$, the inverse of the linear function $p(x) = \frac{2}{5}(x - 3) + 4$.

PRACTICE/HOMEWORK

Use Table 1 to answer questions 1 and 2.

x	y
-1	-2
0	1
1	4
2	7
3	10

TABLE 1

1. Generate the inverse of the linear function.

2. Determine if the inverse is a function. Explain your answer.

Use Table 2 to answer questions 3 and 4.

x	y
-2	7
-1	4
0	3
1	4
2	7

TABLE 2

3. Generate the inverse of the quadratic function.

4. Determine if the inverse is a function. Explain your answer.

Use Table 3 to answer questions 5 and 6.

x	y
-3	-2
-2	-1
-1	0
0	7
1	26

TABLE 3

5. Generate the inverse of the cubic function.

6. Determine if the inverse is a function. Explain your answer.

Use Table 4 to answer questions 7 and 8.

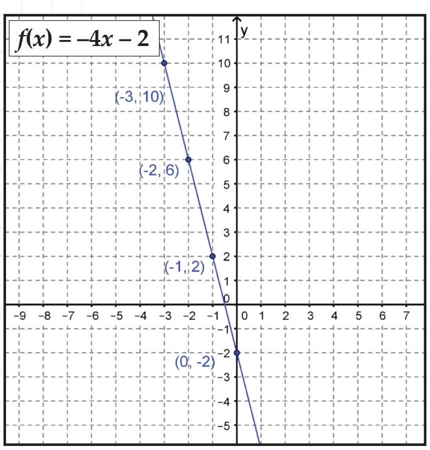
x	y
-1	-2
0	-1
1	1
2	5
3	13

TABLE 4

7. Generate the inverse of the exponential function.

8. Determine if the inverse is a function. Explain your answer.

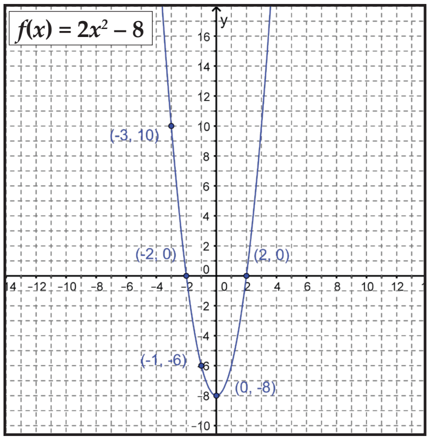
Use the equation and graph below to answer questions 9 and 10.



9. Generate the graph of the inverse of the linear function.

10. Determine the x-intercept(s) and y-intercept(s) of the original function and the x-intercept(s) and y-intercept(s) of the inverse of the function.

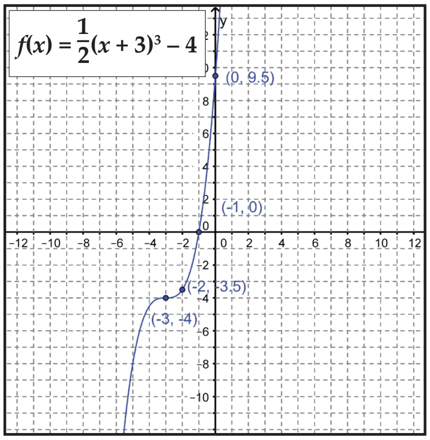
Use the equation and graph below to answer questions 11 and 12.



11. Generate the graph of the inverse of the quadratic function.

12. Determine the x-intercept(s) and y-intercept(s) of the original function and the x-intercept(s) and y-intercept(s) of the inverse of the function.

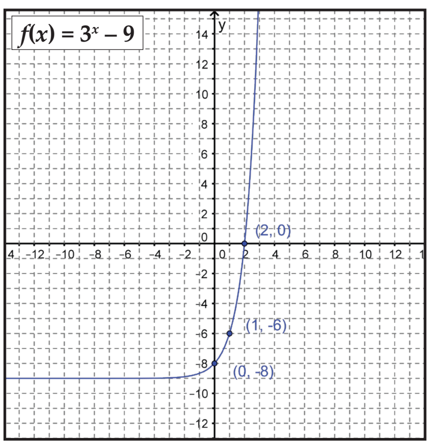
Use the equation and graph below to answer questions 13 and 14.



13. Generate the graph of the inverse of the cubic function.

14. Determine the x-intercept(s) and y-intercept(s) of the original function and the x-intercept(s) and y-intercept(s) of the inverse of the function.

Use the equation and graph below to answer questions 15 and 16.



15. Generate the graph of the inverse of the exponential function.

16. Determine the x-intercept(s) and y-intercept(s) of the original function and the x-intercept(s) and y-intercept(s) of the inverse of the function.

For questions 17 – 20, generate $f^{-1}(x)$, the equation of the inverse of the function given.

17. $f(x) = 3x - 2$

18. $f(x) = \frac{1}{4}(8x + 2)$

19. $f(x) = -2(3x + 1)^2$

20. $f(x) = \frac{1}{2}(2x + 5)^2$

5.6 Factoring Polynomials with Graphs and Tables

FOCUSING QUESTION How can you use tables and graphs to identify the linear factors of polynomial functions?

LEARNING OUTCOMES

- I can factor polynomials using graphs and tables.
- I can use graphs and tables to communicate mathematical ideas and their implications.

ENGAGE

The area of a rectangular painting is 72 square inches. What are some possible dimensions of the painting?

Across the Delaware, Robert Spencer, Wikimedia Commons

EXPLORE

There are relationships among the x-intercepts and factors of polynomial functions. In this section, you will investigate those relationships using graphs and tables.

1. For each of the functions below, use your graphing technology to generate a graph and table. Sketch the graph and record some of the function values in a table like the one shown for each function.

FUNCTION	GRAPH	TABLE		x-INTERCEPT(S)
		x	$f(x)$	
		−4		
		−3		
		−2		
$f(x) = x^2 + x - 6$		−1		
		0		
		1		
		2		
		3		

FUNCTION	GRAPH	TABLE		x-INTERCEPT(S)
$g(x) = x^2 + 5x + 4$		x	$g(x)$	
		−6		
		−5		
		−4		
		−3		
		−2		
		−1		
		0		
		1		
$h(x) = -x^2 + 4x$		x	$h(x)$	
		−2		
		−1		
		0		
		1		
		2		
		3		
		4		
		5		
$j(x) = x^3 - 2x^2 - 5x + 6$		x	$j(x)$	
		−2		
		−1		
		0		
		1		
		2		
		3		
		4		
		5		

Use your graphs and tables to answer the following questions.

2. How did you identify the x-intercepts from the graph or table?

3. For this set of functions, how does the number of x-intercepts compare with the degree of the function?

4. For each x-intercept, $(b, 0)$, of each function, write the binomial $x - b$. Multiply the binomials for each function together and compare the product with the original function.

5. How can you use the x-intercepts from graphs or tables to write the linear factors of a polynomial function?

6. The x-coordinate of an x-intercept of the graph of a function is also called a zero of the function, since that is the x-value that generates a function value of zero. How could you use the zeroes of a function to identify the linear factors?

7. For each of the functions below, use your graphing technology to generate a graph and table. Sketch the graph and record some of the function values in a table like the one shown for each function.

FUNCTION	GRAPH	TABLE		x-INTERCEPT(S)
		x	$f(x)$	
		−0.5		
		0		
		0.5		
$p(x) = 2x^2 - 9x + 4$		1		
		1.5		
		2		
		3		
		4		
		x	$g(x)$	
		−2		
		−1.5		
		−1		
$q(x) = 4x^2 + 4x - 3$		−0.5		
		0		
		0.5		
		1		
		1.5		
		x	$h(x)$	
		−1		
		−0.5		
		0		
$r(x) = 2x^3 - 3x^2 - 5x$		0.5		
		1		
		1.5		
		2		
		2.5		

Use your graphs and tables to answer the following questions.

8. The general form of a quadratic function is $y = ax^2 + bx + c$. When $a = 1$, how does that affect the x-intercepts of the function? When $a \neq 1$, how does that affect at least one x-intercept of the function?

9. Rewrite the coordinates of the non-integer x-intercepts of each function as fractions instead of decimals.

10. For each x-intercept, $(\frac{b}{a}, 0)$, of each function, write the binomial $ax - b$. If the x-coordinate of the x-intercept is an integer, let $a = 1$. Multiply the binomials for each function together and compare the product with the original function.

11. Identify the x-intercepts and factors of $m(x) = x^2 - 6x + 9$. How is the graph of $m(x)$ different from the graphs of the other quadratic functions you factored?

REFLECT

- **If you can identify the *x*-intercepts or zeroes of a polynomial function from a graph or a table, how can you write the factors of the polynomial function?**
- **What does the degree of a polynomial function tell you about the number of factors to expect?**

ELPS FEATURE

- **Retell or summarize, using increasingly complex English, how the *x*-intercepts of a polynomial function, the zeros of the function, and the linear factors of the polynomial function are related.**

EXPLAIN

Factoring a polynomial is a method of rewriting a polynomial as a set of factors that can be multiplied together in order to generate the original polynomial. Writing a polynomial in factored form allows you to do several things, including divide polynomials, simplify rational or polynomial expressions, and look for solutions to polynomial equations.

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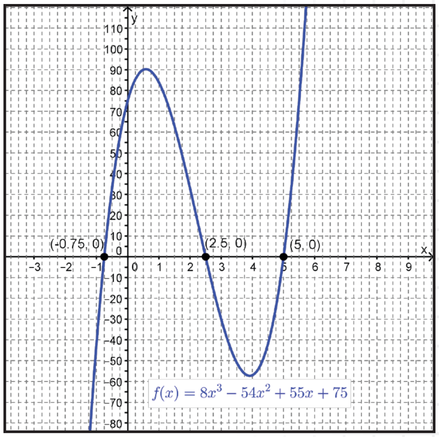
General Form $f(x) = 3x^2 + 2x - 8$ **Polynomial Form** $\leftrightarrow$ **Factored Form** $f(x) = (3x - 4)(x + 2)$

The factors of a polynomial function are related to the x-intercepts or zeroes of the function. For example, the graph of $f(x) = 8x^3 - 54x^2 + 55x + 75$ with its x-intercepts is shown.



The function $f(x)$ has three x-intercepts labeled in the graph: $(-0.75, 0)$, $(2.5, 0)$, and $(5, 0)$. Write the non-integer x-intercepts as fractions: $\left(-\frac{3}{4}, 0\right)$, $\left(\frac{5}{2}, 0\right)$ and $(5, 0)$. From here, the zeros of $f(x)$ can be identified as $-\frac{3}{4}$, $\frac{5}{2}$, and 5. The zeros can be used to identify the linear factors of $f(x)$ so that the function can be rewritten in factored form. If the zero is a fraction $\frac{b}{a}$, then the linear factor is the binomial $ax - b$.

$-\frac{3}{4} \rightarrow (4x - (-3)) = 4x + 3$ $\qquad$ $\frac{5}{2} \rightarrow (2x - 5) = 2x - 5$ $\qquad$ $5 \rightarrow (x - 5) = x - 5$

The function $f(x)$ can be rewritten as the product of these linear factors.

$$f(x) = (4x + 3)(2x - 5)(x - 5)$$

WHY DOES THIS WORK?

Recall that when you are multiplying a set of numbers together, if one of the numbers is 0, then the entire product will be zero.

$$3.5 \times 1{,}376 \times 0 \times 14\tfrac{7}{12} = 0$$

For the function $f(x)$, if any one of the three factors is equal to 0, then the entire product, or function value, will also be 0. When the function value is 0 (i.e., $f(x) = 0$), the graph shows an x-intercept. Thus, the x-intercepts identify the x-values that will generate a function value of 0.

If you have an x-intercept at $\left(\frac{b}{a}, 0\right)$, then $x = \frac{b}{a}$. Use this relationship to write a linear expression.

$$\begin{aligned} x &= \frac{b}{a} \\ ax &= b \\ ax - b &= 0 \end{aligned}$$

With our original function, $f(x)$, we know that there is an x-intercept at $\left(\frac{5}{2}, 0\right)$. Set $x = \frac{5}{2}$ and write the linear factor associated with this x-intercept.

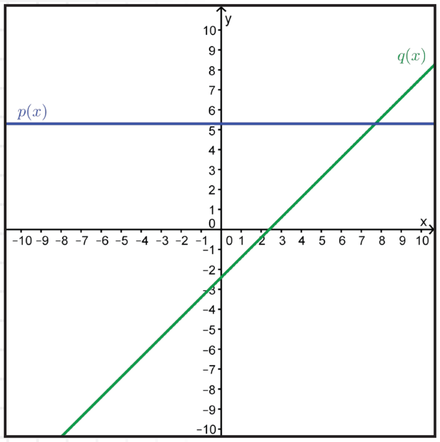
$$\begin{aligned} x &= \frac{5}{2} \\ 2x &= 5 \\ 2x - 5 &= 0 \end{aligned}$$

x	$f(x)$
−1	−42
−0.75	0
−0.5	33
0	75
1	84
2	33
2.5	0
3	−30
4	−57
5	0
6	189

You can also use a table of values to identify the zeros (x-intercepts) of a function. The table shows a set of function values for $f(x) = 8x^3 - 54x^2 + 55x + 75$.

The zeros of $f(x)$ are the x-values where $f(x) = 0$. In the table, there are three x-values where $f(x) = 0$: $x = -0.75$, 2.5, and 5.

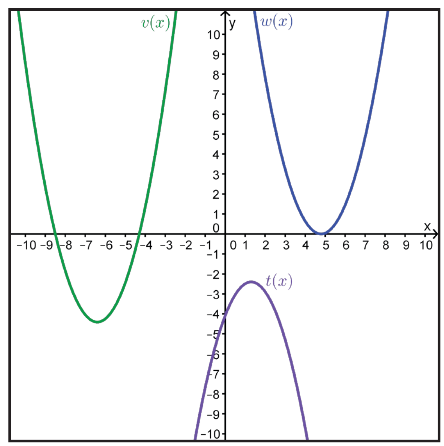
Once the zeros have been identified, write any non-integer zeros as fractions and use the fraction to write a linear factor.



FACTORS AND POLYNOMIAL DEGREE

There is also a relationship between the degree of a polynomial and the number of factors you can expect.

A linear function is a polynomial function of degree one. A linear function has up to one x-intercept, but it may have no x-intercepts. The function $q(x) = ax + b$, where $a \neq 0$, has one x-intercept at $\left(\frac{b}{a}, 0\right)$. However, the constant function $p(x) = b$ is a horizontal line with no x-intercept.

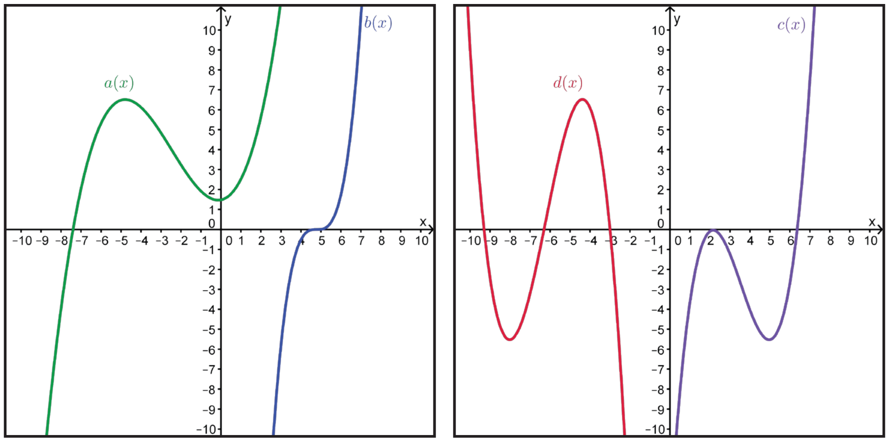


A quadratic function is a polynomial function of degree two. A quadratic function has up to two x-intercepts, but it may have only one x-intercept or no x-intercepts.

- The function $v(x)$ has two x-intercepts because the function crosses the x-axis.
- The function $w(x)$ has only one x-intercept because the function touches the x-axis but does not cross the x-axis.
- The function $t(x)$ has no x-intercepts because it does not touch or cross the x-axis.

A cubic function is a polynomial function of degree three. A cubic function has up to three x-intercepts, but it may have fewer than three x-intercepts.

- The functions $a(x)$ and $b(x)$ each have only one x-intercept. The cubic function crosses the x-axis only once.
- The function $c(x)$ has two x-intercepts. The graph of the function touches the x-axis, decreases, and then increases to cross the x-axis.
- The function $d(x)$ has three x-intercepts. The graph of the function crosses the x-axis in three separate locations.



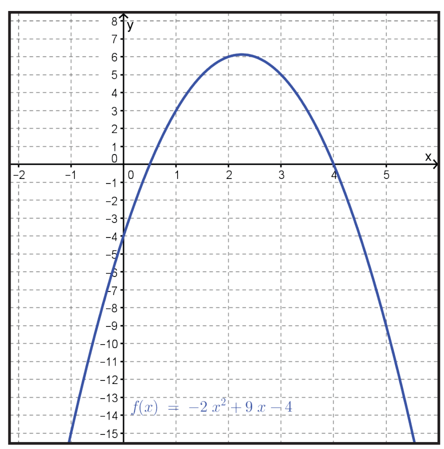
FACTORING POLYNOMIAL FUNCTIONS WITH GRAPHS AND TABLES

Polynomial functions can be factored, or written as a product of linear factors, by identifying their *x*-intercepts from a graph or their zeroes from a table.

- In a graph, locate an *x*-intercept, $\left(\frac{b}{a}, 0\right)$, by identifying the exact location where the graph touches or crosses the *x*-axis. The linear factor associated with this *x*-intercept is $ax - b$.
- In a table, locate a zero, $\frac{b}{a}$, by determining the *x*-value that generates a function value of 0. The linear factor associated with this zero is $ax - b$.

EXAMPLE 1

Factor the quadratic function $f(x) = -2x^2 + 9x - 4$ using a graph.



STEP 1 **Determine the *x*-intercepts of the quadratic function from its graph. Verify your answers by using the equation of the function.**

The graph of $f(x)$ appears to cross the *x*-axis at $\frac{1}{2}$ and 4.

$f(x) = -2x^2 + 9x - 4$

$f\left(\frac{1}{2}\right) = -2\left(\frac{1}{2}\right)^2 + 9\left(\frac{1}{2}\right) - 4 = -2\left(\frac{1}{4}\right) + \frac{9}{2} - 4 = -\frac{1}{2} + \frac{9}{2} - 4 = 4 - 4 = 0$

$f(4) = -2(4)^2 + 9(4) - 4 = -2(16) + 36 - 4 = -32 + 36 - 4 = 4 - 4 = 0$

Therefore, both $\frac{1}{2}$ and 4 are zeroes of $f(x)$.

STEP 2 **Use each *x*-intercept to determine binomial factors.**

$$x = \frac{1}{2} \qquad\qquad\qquad$$
$$2x = 1 \qquad \text{and} \qquad x = 4$$
$$2x - 1 = 0 \qquad\qquad x - 4 = 0$$

The binomial factors of $f(x)$ are $(2x - 1)$ and $(x - 4)$.

STEP 3 **Multiply the binomial factors and compare to the symbolic representation of $f(x)$.**

$$(2x - 1)(x - 4) = 2x^2 - 8x - x + 4 = 2x^2 - 9x + 4 = -(-2x^2 + 9x - 4) = -f(x)$$

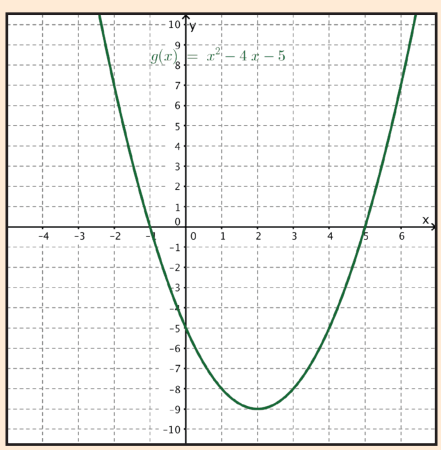
Because the parabola opens downward, the value of the parameter a is negative. The binomial factors do not take this into account. Therefore, you must multiply the function by −1 in order to ensure that $a < 0$ in the factored form of the function.

STEP 4 **Write the function as a product of its binomial factors. Since the parabola opens downward, multiply by −1.**

$$-f(x) = (2x - 1)(x - 4)$$
$$(-1)(-f(x)) = (-1)(2x - 1)(x - 4)$$
$$f(x) = -(2x - 1)(x - 4)$$

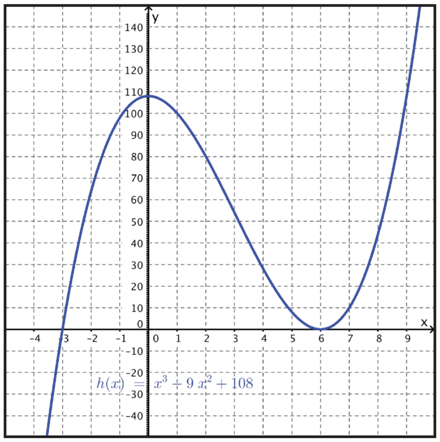
YOU TRY IT! #1

Factor the quadratic function $g(x) = x^2 - 4x - 5$ using a graph.



EXAMPLE 2

Factor the cubic function $h(x) = x^3 - 9x^2 + 108$ using a graph.



STEP 1 **Determine the x-intercepts of the cubic function from its graph. Verify your answers by using the equation of the function.**

The graph of $h(x)$ appears to cross the x-axis at -3 and touch the x-axis at 6.

$h(x) = x^3 - 9x^2 + 108$

$h(-3) = (-3)^3 - 9(-3)^2 + 108 = -27 - 9(9) + 108 = -27 - 81 + 108 = -108 + 108 = 0$

$h(6) = (6)^3 - 9(6)^2 + 108 = 216 - 9(36) + 108 = 216 - 324 + 108 = -108 + 108 = 0$

Therefore, both -3 and 6 are zeroes of $h(x)$.

STEP 2 **Use each x-intercept to determine binomial factor(s).**

$x = -3$ and $x = 6$

$x + 3 = 0$ $\quad$ $x - 6 = 0$

The binomial factors of $h(x)$ are $(x + 3)$ and $(x - 6)$.

Because the graph touches the x-axis at 6 rather than crossing it, the binomial factor $(x - 6)$ is repeated.

STEP 3 **Multiply the binomial factors and compare to the symbolic representation of $h(x)$.**

$(x + 3)(x - 6)(x - 6) = (x^2 - 6x + 3x - 18)(x - 6) = (x^2 - 3x - 18)(x - 6)$

$= x^3 - 6x^2 - 3x^2 + 18x - 18x + 108 = x^3 - 9x^2 + 108 = h(x)$

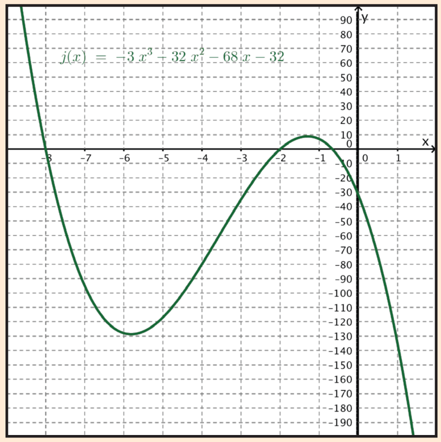
STEP 4 **Write the function as a product of its binomial factors.**

$h(x) = (x + 3)(x - 6)(x - 6)$

$h(x) = (x + 3)(x - 6)^2$

YOU TRY IT! #2

Factor the cubic function $j(x) = -3x^3 - 32x^2 - 68x - 32$ using a graph.



EXAMPLE 3

Factor the quadratic function $p(x) = 16x^2 - 56x + 49$ using a table.

x	$p(x)$
1	9
1.25	4
1.5	1
1.75	0
2	1
2.25	4
2.5	9
2.75	16
3	25

STEP 1 **Determine the x-intercepts of the quadratic function from its table.**

The table for $p(x)$ has a function value of zero at 1.75, or $\frac{7}{4}$. Therefore, $\frac{7}{4}$ is a zero of $p(x)$.

STEP 2 **Use each x-intercept to determine binomial factor(s).**

$$x = \frac{7}{4}$$
$$4x = 7$$
$$4x - 7 = 0$$

The binomial factor of $p(x)$ is $(4x - 7)$.

As can be seen from the values in the table, the point (1.75, 0) is the vertex of the parabola, since it is the minimum value in the table and the values around it demonstrate that $x = 1.75$ is the line of symmetry of the parabola.

MINIMUM → 1.75

x	$p(x)$
1	9
1.25	4
1.5	1
1.75	0
2	1
2.25	4
2.5	9
2.75	16
3	25

Because the parabola touches the x-axis at $\frac{7}{4}$ rather than crossing it, the binomial factor $(4x - 7)$ is repeated.

STEP 3 **Multiply the binomial factors and compare to the symbolic representation of $p(x)$.**

$$(4x - 7)(4x - 7) = 16x^2 - 28x - 28x + 49 = 16x^2 - 56x + 49 = p(x)$$

STEP 4 **Write the function as a product of its binomial factors.**

$$p(x) = (4x - 7)(4x - 7)$$
$$p(x) = (4x - 7)^2$$

YOU TRY IT! #3

Factor the quadratic function $r(x) = x^2 + 3x + 4$ using a table.

x	$r(x)$
-3	4
-2.5	2.75
-2	2
-1.5	1.75
-1	2
-0.5	2.75
0	4
0.5	5.75
1	8

EXAMPLE 4

x	$t(x)$
-5	116
-4	0
-3	-40
-2	-34
-1	-12
-0.5	-4.375
0	-4
0.5	-14.625
1	-40

Factor the cubic function $t(x) = -5x^3 - 22x^2 - 9x - 4$ using a table. The table of values shows all x-intercepts of $t(x)$.

STEP 1 **Determine the x-intercepts of the cubic function from the table. Verify your answers by using the equation of the function.**

The graph of $t(x)$ appears to cross the x-axis at -4. There does not appear to be any other zero since the function values decrease to approximately -40, then increase to approximately -4 and then decrease again without crossing the x-axis again.

Therefore, -4 is a zero of $t(x)$.

STEP 2 **Use each x-intercept to determine binomial factor(s).**

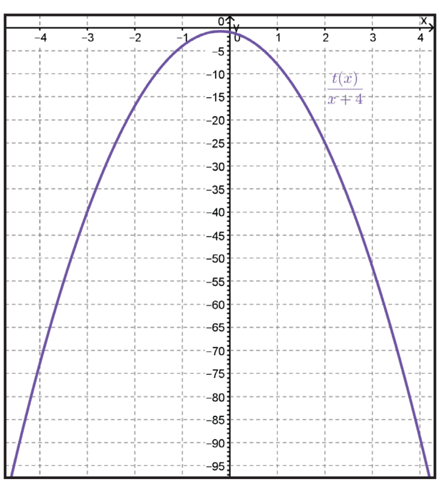
$$x = -4$$
$$x + 4 = 0$$

The binomial factor of $t(x)$ is $(x + 4)$. Since $t(x)$ is a cubic function (function of degree three) and it has only one binomial factor that is not repeated, then its other factor must be quadratic (function of degree two).

STEP 3 **Use graphing technology to graph or generate a table for the function $\frac{t(x)}{(x + 4)}$.**

Use the finite differences in the table or your knowledge of function transformations in the graph to determine the quadratic factor of $t(x)$.

Using the patterns in the finite differences in the table, you can determine that the quadratic factor is $(-5x^2 - 2x - 1)$.



x	$\frac{t(x)}{x + 4}$
-3	-40
-2	-17
-1	-4
0	-1
1	-8
2	-25
3	-52

Notice that this function, $y = -5x^2 - 2x - 1$ has no real number linear factors. The graph of the function does not touch or cross the x-axis and the table values show no place where the function values change from positive to negative or vice versa.

STEP 3 **Multiply the binomial and quadratic factors and compare to the symbolic representation of $t(x)$.**

$(x + 4)(-5x^2 - 2x - 1) = -5x^3 - 2x^2 - x - 20x^2 - 8x - 4$
$= -5x^3 - 2x^2 - 20x^2 - x - 8x - 4 = -5x^3 - 22x^2 - 9x - 4 = t(x)$

STEP 4 **Write the function as a product of its binomial and quadratic factors.**

$t(x) = (x + 4)(-5x^2 - 2x - 1)$ or $t(x) = -(x + 4)(5x^2 + 2x + 1)$
Either is correct.

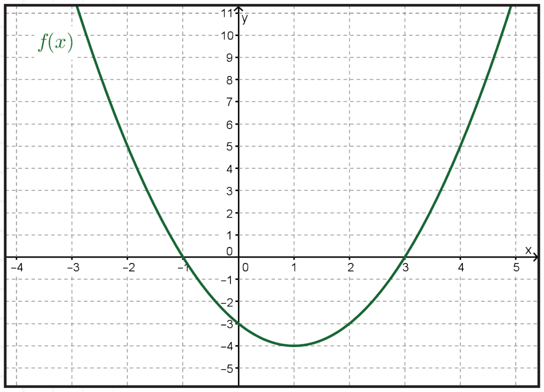
YOU TRY IT! #4

Factor the cubic function $w(x) = x^3 - 6x^2 - 9x + 14$ using a table.

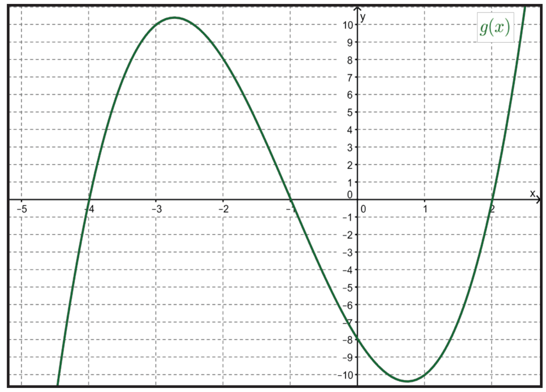
x	$w(x)$
-4	-110
-3	-40
– 2	0
-1	16
1	0
2	-20
3	-40
4	-54
5	-56
6	-40
7	0
8	70

PRACTICE/HOMEWORK

1. Determine the x-intercepts of $f(x)$.



2. Determine the zeros of $g(x)$.



For questions 3 – 6, determine the factors of the polynomial function given the x-intercepts of the function.

3. The x-intercepts of a quadratic function, $f(x)$, are (5, 0) and (1, 0). Determine the factors of the function.

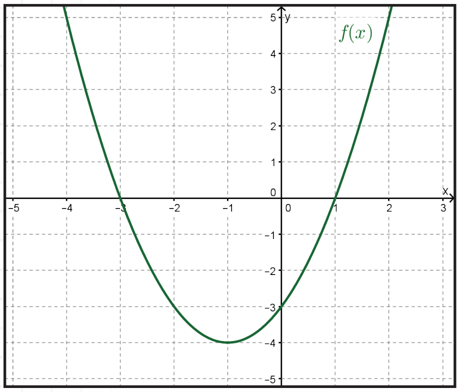
4. The x-intercepts of a quadratic function, $p(x)$, are (-2.5, 0) and (3, 0). Determine the factors of the function.

5. The x-intercepts of a cubic function, $f(x)$, are (6, 0), (-2, 0) and (0.5, 0). Determine the factors of the function.

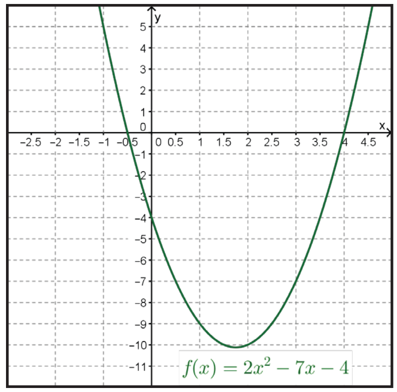
6. The x-intercepts of a cubic function, $p(x)$, are (-3, 0), and (1, 0) and (3, 0). Determine the factors of the function.

For questions 7 – 10, identify the x-intercept(s) of each function, and then write the quadratic function in factored form.

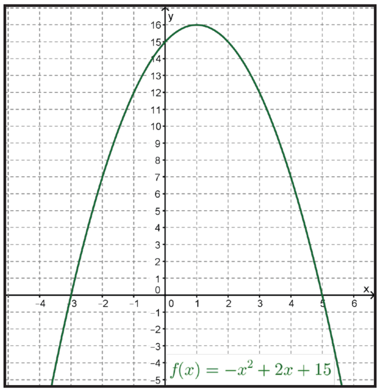
7. $f(x) = x^2 + 2x - 3$



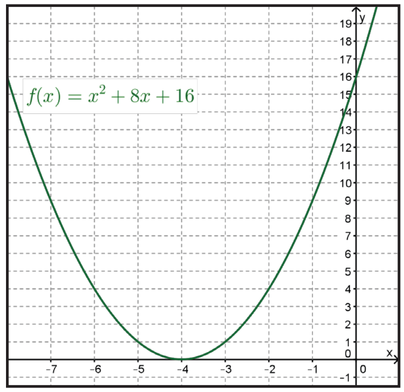
8. $f(x) = 2x^2 - 7x - 4$



9. $f(x) = -x^2 + 2x + 15$

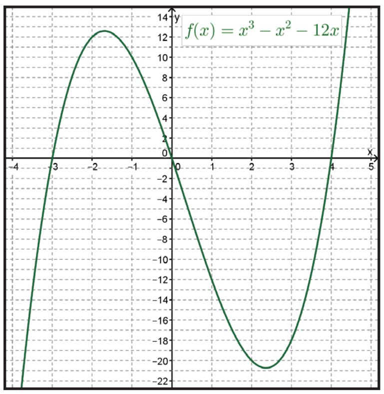


10. $f(x) = x^2 + 8x + 16$

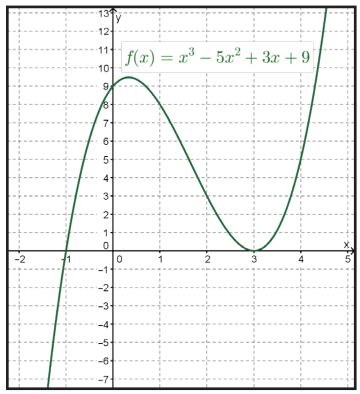


For questions 11 – 14, identify the x-intercept(s) of each function, and then write the cubic function in factored form.

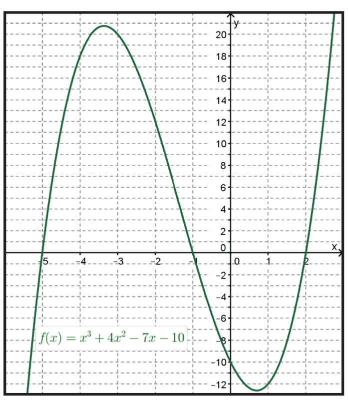
11. $f(x) = x^3 - x^2 - 12x$



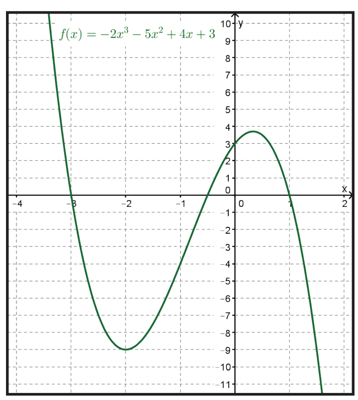
12. $f(x) = x^3 - 5x^2 + 3x + 9$



13. $f(x) = x^3 + 4x^2 - 7x - 10$



14. $f(x) = -2x^3 - 5x^2 + 4x + 3$



For questions 15 – 17 identify the zeros of each function, and then write the quadratic function in factored form.

15.

x	$p(x)$
−6	9
−5	4
−4	1
−3	0
−2	1
−1	4

16.

x	$p(x)$
−2	0
−1	4
0	6
1	6
2	4
3	0
4	−6

17.

x	$p(x)$
−3	7
−2.5	3
−2	0
−1.5	−2
−1	−3
−0.5	−3
0	−2
0.5	0
1	3

For questions 18 – 19 identify the zeros of each function, then write the cubic function in factored form.

18.

x	$q(x)$
-4	-10
-3	0
-2	0
-1	-4
0	-6
1	0

19.

x	$q(x)$
-4	15
-3	0
-2	-3
-1	0
0	3
1	0
2	-15

For questions 20 – 23, use your graphing technology to generate a table or a graph of the given function. Identify what type of function it is (quadratic or cubic) and the x-intercepts of the function. Write the function as a product of its factors.

20. $g(x) = -x^3 - 3x^2 + 6x + 8$

21. $g(x) = -x^2 - 4x - 4$

22. $g(x) = x^2 - 10x + 21$

23. $q(x) = 4x^3 - 13x^2 + 3x$

6.6 Solving Systems of Three Linear Equations

FOCUSING QUESTION How can I use matrices and technology to represent and solve a system of three linear equations?

LEARNING OUTCOMES

- I can represent a problem with a system of three linear equations using matrices and technology.
- I can use matrices and technology to solve a problem involving a system of three linear equations.
- I can select tools, including paper and pencil and technology, as appropriate to solve

ENGAGE

Lashondra owns a Christmas tree farm. She wants to plant both Douglas fir trees and Scotch pine trees to sell next Christmas. Douglas fir trees cost \$250 per acre to plant and Scotch pine trees cost \$175 per acre to plant. Lashondra will plant 50 acres of trees and will spend \$11,000. How many acres of each tree will Lashondra plant? Define your variables, write a system of equations, and represent the system using a matrix equation.

EXPLORE

Image source: Pixabay

Gulf Stream Lumber has a plant with three sawmills, A, B, and C. If all three sawmills run all day, then the three sawmills produce 5,700 board-feet of lumber. If sawmill A runs for two days and sawmill B runs for one day, they produce 4,900 board-feet of lumber. If only sawmills B and C run all day, then the two sawmills together produce 4,200 board-feet of lumber.

1. Write a system of equations that you could use to represent this problem. Let the variables a, b, and c represent the amount of lumber produced in one day by their respective sawmills.

2. Use the coefficients of a, b, and c to write matrix A where each row represents one equation and each column represents one of the variables a, b, and c. If an equation does not contain all three variables, be sure to use a term with 0 as a coefficient as a placeholder.

3. Use the constants from each equation to write matrix B where each row represents one equation.

4. Use the matrix $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to represent the variables and write a matrix equation relating matrix A, matrix B, and the variable matrix.

The inverse of a 3 × 3 matrix can be calculated using the formula shown.

$$\text{If } A = \begin{bmatrix} a & b & c \\ d & f & g \\ h & j & k \end{bmatrix}, \text{ then } A^{-1} = \frac{1}{a(fk-gj)-b(dk-gh)-c(dj-fh)} \begin{bmatrix} fk-gj & cj-bk & bg-cf \\ gh-dk & ak-ch & cd-ag \\ dj-fh & bh-aj & af-bd \end{bmatrix}.$$

5. Use the inverse formula with paper and pencil, graphing technology (e.g., calculator or app), or an online matrix inverse calculator to calculate the matrix A^{-1}.

6. Use the matrix A^{-1} and the matrix equation from a previous question to solve for $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

7. Write the solution to the system and explain what it means in the context of the original problem.

Repetivive computations such as those involved with calculating the inverse of a 3 × 3 matrix are easily programmed into a computer. Coding algorithms such as calculating the inverse of a matrix have many applications for computer programming.

Use the situation below to answer questions 8 – 10.

A candy company packages chocolate gift boxes using white chocolate, milk chocolate, and dark chocolate.

- A box of 1 package of white chocolate, 3 packages of milk chocolate, and 3 packages of dark chocolate weighs 14 ounces.
- A box of 1 package of white chocolate, 3 packages of milk chocolate, and 4 packages of dark chocolate weighs 17 ounces.
- A box of 1 package of white chocolate, 4 packages of milk chocolate, and 3 packages of dark chocolate weighs 15 ounces.

8. Let x represent the weight of one package of white chocolate, y represent the weight of one package of milk chocolate, and z represent the weight of one package of dark chocolate. Write a system of three linear equations that you can use to represent this problem.

9. Use technology to represent the system of three linear equations.

10. Use technology to solve the system of three linear equations. Interpret the solution within the context of the problem.

REFLECT

- **When solving a system of three linear equations using matrices, is it easier to use paper and pencil or technology to solve the problem? Explain your reasoning.**

- **How is the idea of left-multiplying by the inverse matrix to solve a matrix equation related to multiplying by the inverse to solve an equation with real numbers (e.g., to solve $2x = 7$, multiply by the inverse of 2)?**

EXPLAIN

Situations with three unknowns require particular pieces of information in order to solve for the values of those unknowns. To write a system of equations for the situation, you need to have as many equations as you do unknowns in order to solve the system. If the equations are all linear equations, then you can use matrices to solve the system of three linear equations.

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ELPS ACTIVITY

Read the material in this section with a partner. Use support from your peer by asking each other the following questions.

- Could you please summarize what this paragraph is saying?
- What word that you know is similar to this word?
- Where have you seen something like this before?

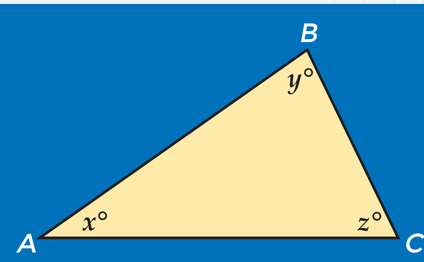
REPRESENTING A SYSTEM OF THREE LINEAR EQUATIONS USING MATRICES

For a system of three linear equations with three unknowns, you can use a matrix equation with 3×3 matrices to represent the system. This process is similar to what you did with systems of two linear equations with two unknowns.

- Make sure that all linear equations are in standard form, $Ax + By + Cz = D$. If one variable is missing from the equation, be sure to use a term with 0 as the coefficient as a placeholder.

- Place the coefficients of the unknowns into a 3 × 3 coefficient matrix.
- Place the unknowns into a 3 × 1 variable matrix.
- Place the constants (D when the equation is in standard form) into a 3 × 1 constant matrix.

Consider the triangle problem shown. If x, y, and z each represent the measure of one interior angle of ΔABC, then you can write the system of three linear equations shown.



In ΔABC, the sum of the measures of the interior angles is 180°. Also, $m\angle B$ is twice $m\angle A$ and the sum of $m\angle A$ and $m\angle B$ is twice $m\angle C$. What are the measures of each interior angle of ΔABC?

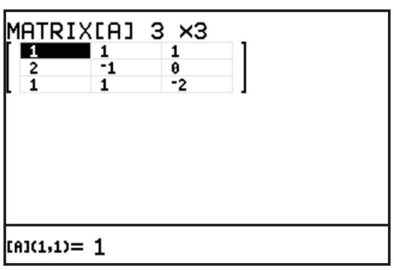
$$\begin{cases} x + y + z = 180 \\ y = 2x \\ x + y = 2z \end{cases} \longrightarrow \begin{cases} x + y + z = 180 \\ 2x - y + 0z = 0 \\ x + y - 2z = 0 \end{cases} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 180 \\ 0 \\ 0 \end{bmatrix}$$

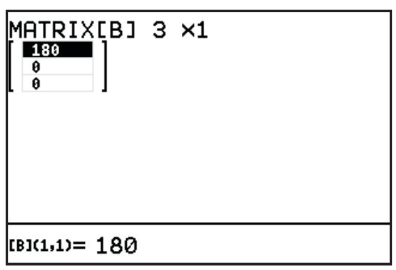
coefficient matrix

variable matrix

constant matrix

Once you have the system of three linear equations represented in a matrix, you can use technology to represent the system. In a graphing calculator, you enter the numeric values of each matrix entry into a matrix app.





DETERMINING THE INVERSE OF A 3 × 3 MATRIX

As you have seen with systems of two linear equations, in order to solve the matrix equation for the variable matrix, you need to left-multiply by the inverse of the coefficient matrix. In this case, the coefficient matrix is a 3 × 3 matrix.

For a 3 × 3 matrix, $A = \begin{bmatrix} a & b & c \\ d & f & g \\ h & j & k \end{bmatrix}$, the product of the inverse of matrix A, which is written as A^{-1}, and matrix A must be equal to the identity matrix.

$$A^{-1} \times \begin{bmatrix} a & b & c \\ d & f & g \\ h & j & k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving this matrix equation for A^{-1} generates a formula for determining A^{-1} from a given matrix A.

If $A = \begin{bmatrix} a & b & c \\ d & f & g \\ h & j & k \end{bmatrix}$, then $A^{-1} = \frac{1}{a(fk-gj)-b(dk-gh)-c(dj-fh)} \begin{bmatrix} fj-gj & cj-bk & bg-cf \\ gh-dk & ak-ch & cd-ag \\ dj-fh & bh-aj & af-bd \end{bmatrix}$.

You can certainly calculate the inverse of a 3 × 3 matrix using paper and pencil and a calculator by hand. Or, you can use graphing technology, an app, or an online matrix inverse calculator to calculate the inverse of the coefficient matrix.

SOLVING A SYSTEM OF THREE LINEAR EQUATIONS USING MATRICES

Once you have written your matrix equation, you can use the inverse of the coefficient matrix to solve for the variable matrix. Remember that matrix multiplication is not commutative. So if you want the product of $[A]^{-1}$ and $[A]$ to equal the identity matrix, then you will need to left-multiply $[A]^{-1}$ by $[A]$.

$$[A]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [B] \longrightarrow [A]^{-1}[A]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [A]^{-1}[B]$$

With the triangle problem, you represented the situation with a system of three linear equations and then used the equations to create a matrix equation. You also used technology to represent the system of equations through matrices. Now, you can calculate the inverse of the coefficient matrix and begin multiplication.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 180 \\ 0 \\ 0 \end{bmatrix}$$

The inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$ can be calculated using either the inverse formula or technology.

$$[A]^{-1} = \begin{bmatrix} \frac{2}{9} & \frac{1}{3} & \frac{1}{9} \\ \frac{4}{9} & -\frac{1}{3} & \frac{2}{9} \\ \frac{1}{3} & 0 & -\frac{1}{3} \end{bmatrix}$$

Use $[A]^{-1}$ to solve the original matrix equation for the variable matrix, $\begin{bmatrix} x \\ y \\ Z \end{bmatrix}$.

$$[A]^{-1}[A]\begin{bmatrix} x \\ y \\ Z \end{bmatrix} = [A]^{-1}[B]$$

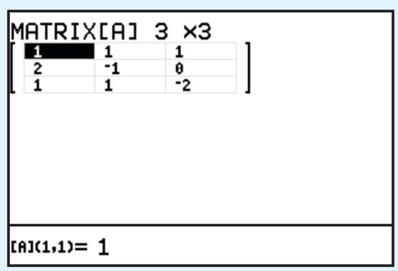
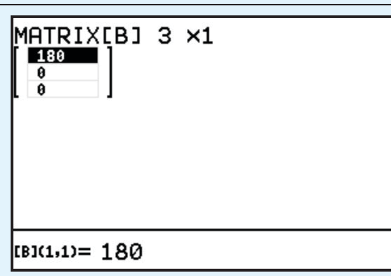
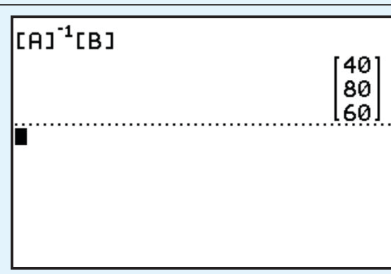
$$\begin{bmatrix} \frac{2}{9} & \frac{1}{3} & \frac{1}{9} \\ \frac{4}{9} & -\frac{1}{3} & \frac{2}{9} \\ \frac{1}{3} & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & \frac{1}{3} & \frac{1}{9} \\ \frac{4}{9} & -\frac{1}{3} & \frac{2}{9} \\ \frac{1}{3} & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 180 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 40 \\ 80 \\ 60 \end{bmatrix}$$

The solution to the given system is (40, 80, 60). In the context of the problem, the measures of the three interior angles of ΔABC are 40°, 80°, and 60°.

USING TECHNOLOGY TO SOLVE SYSTEMS OF THREE LINEAR EQUATIONS WITH MATRICES

Graphing technology, such as a graphing calculator or app, can be extremely beneficial when solving systems of equations with matrices.

Write the system of linear equations as a matrix equation. Make sure that all three linear equations are in standard form.	$\begin{cases} x+y+z=180 \\ y=2x \\ x+y=2z \end{cases} \rightarrow \begin{cases} x+y+z=180 \\ 2x-y+0z=0 \\ x+y-2z=0 \end{cases}$ $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 180 \\ 0 \\ 0 \end{bmatrix}$
Enter the coefficient matrix into one matrix of your graphing technology.	
Enter the constant matrix into a second matrix of your graphing technology.	
Use matrix operations to calculate $[A]^{-1}[B]$.	

SYSTEMS OF THREE LINEAR EQUATIONS WITH MATRICES

Matrices can be used to represent and solve systems of three linear equations.

- Make sure that all three linear equations are in standard form, $Ax + By + Cz = D$, where A, B, C, and D are integers and at least one of A, B, and C is not equal to 0.
- Use the matrix equation $[A]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [B]$ where $[A]$ represents the coefficient matrix and $[B]$ represents the constant matrix.
- Determine the inverse of the coefficient matrix, $[A]^{-1}$.
- Left-multiply both sides of the matrix equation by $[A]^{-1}$. The left member of the equation, $[A]^{-1}[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, is the identity matrix.
- Technology can be used to enter and calculate the values of the variable matrix, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [A]^{-1}[B]$.

EXAMPLE 1

Points X and Y are between points W and Z. The distance between points W and Z is thirty-six millimeters. The distance between points W and X is half the distance from point X to point Y. The distance between points Y and Z is one millimeter less than the distance from point X to point Y. What is the length in millimeters of each segment? Write a system of three linear equations to represent the situation and a matrix equation to represent the system. Represent and solve the system using matrices with technology.

STEP 1 **Define variables to represent the unknowns and use them write a system of linear equations to represent the system.**

Since the problem asks you to determine the length of the segments, the variables will represent the lengths, in millimeters, of the three segments. Let x represent the length of $\overline{WX}$, let y represent the length of $\overline{XY}$, and let z represent the length of $\overline{YZ}$.

"The segment shown is a total of thirty-six millimeters long." $\rightarrow x + y + z = 36$.
"The distance between points W and X is half the distance from point X to point Y." $\rightarrow x = \frac{1}{2}y$.
"The distance between points Y and Z is one millimeter less than the distance from point X to point Y." $\rightarrow z = y - 1$.

Therefore, a system of linear equations that represents the situation is

$$\begin{cases} x + y + z = 36 \\ x = \frac{1}{2}y \\ z = y - 1 \end{cases}$$

STEP 2 **Write a matrix equation that corresponds to the system of linear equations you wrote in Step 1. If necessary, rewrite any linear equations that are not already in standard form before writing the matrix equation.**

Two of the linear equations in the system you wrote in Step 1 are not in standard form, so it is necessary to rewrite both equations. Remember that if an equation does not contain all three variables, you should use a term with 0 as a coefficient as a placeholder.

$$\begin{cases} x + y + z = 36 \\ x = \frac{1}{2}y \\ z = y - 1 \end{cases} \rightarrow \begin{cases} x + y + z = 36 \\ x - \frac{1}{2}y + 0z = 0 \\ 0x - y + z = -1 \end{cases}$$

Although some of the variables appear to have no coefficients, remember that a coefficient of one is implied. A matrix equation that represents the system above is

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{\text{coefficient matrix}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\text{variable matrix}} = \underbrace{\begin{bmatrix} 36 \\ 0 \\ -1 \end{bmatrix}}_{\text{constant matrix}}.$$

STEP 3 **Represent and solve the system using matrices with technology.**

To represent the system using technology, enter a 3×3 matrix A with entries equal to the coefficient matrix and a 3×1 matrix B with entries equal to the constant matrix.

To solve the system using technology, multiply the inverse of matrix A by matrix B.

$$A^{-1}B = \begin{bmatrix} 7.4 \\ 14.8 \\ 13.8 \end{bmatrix}$$

$$\begin{cases} x + y + z = 36 \\ x = \frac{1}{2}y \\ z = y - 1 \end{cases}$$

is the system of linear equations that represents this situation if x is the length in millimeters of $\overline{WX}$, y is the length in millimeters of $\overline{XY}$, and z is the length in millimeters of $\overline{YZ}$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 36 \\ 0 \\ -1 \end{bmatrix}$$

is the matrix equation that represents the system of equations.

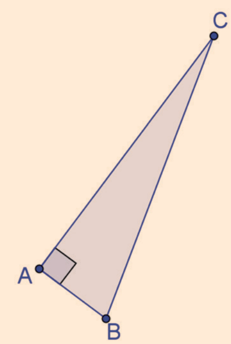
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7.4 \\ 14.8 \\ 13.8 \end{bmatrix}$$

is the solution for the system determined using matrices and technology.

The length of $\overline{WX}$ is 7.4 mm, the length of $\overline{XY}$ is 14.8 mm, and the length of $\overline{YZ}$ is 13.8 mm.

YOU TRY IT! #1

The perimeter of the right triangle shown is six hundred fifty centimeters. The length of its hypotenuse is one centimeter more than the length of its longer leg. The length of the longer leg of the right triangle is sixty-three centimeters less than fifteen times the length of its shorter leg. How long is each side of the right triangle? Write a system of three linear equations to represent the situation and a matrix equation to represent the system. Represent and solve the system using matrices with technology.



EXAMPLE 2

A baseball stadium has three levels of seats: field level, mezzanine level, and upper level. Field level seat tickets sell for \$29 each. Mezzanine level seat tickets sell for \$19.99 apiece. Upper level seat tickets sell for \$13.50. There are as many field level seats as there are mezzanine and upper level seats combined. The stadium has 58,000 seats and takes in \$1,336,340 for every sold out game. How many seats are in each level of the baseball stadium? Write a system of three linear equations to represent the situation and a matrix equation to represent the system. Represent and solve the system using matrices with technology.

STEP 1 **Define variables to represent the unknowns and use them write a system of linear equations to represent the system.**

Since the problem asks you to determine how many seats are in each level of the baseball stadium, the variables will represent the number of seats in each level. Let f represent the number of seats in the field level, let m represent the number of seats in the mezzanine level, and let u represent the number of seats in the upper level of the baseball stadium.

"There are as many field level seats as there are mezzanine and upper level seats combined." $\rightarrow f = m + u$.
"The stadium has 58,000 seats...." $\rightarrow f + m + u = 58{,}000$.
"Field level seat tickets sell for \$29 each. Mezzanine level seat tickets sell for \$19.99 apiece. Upper level seat ticket prices are \$13.50. …and takes in \$1,336,340 for every sold out game" $\rightarrow 29f + 19.99m + 13.50u = 1{,}336{,}340$.

Therefore, a system of linear equations that represents the situation is

$$\begin{cases} f = m + u \\ f + m + u = 58{,}000 \\ 29f + 19.99m + 13.50u = 1{,}336{,}340 \end{cases}$$

Step 2 **Write a matrix equation that corresponds to the system of linear equations you wrote in Step 1. If necessary, rewrite any linear equations that are not already in standard form before writing the matrix equation.**

One of the linear equations in the system you wrote in Step 1 is not in standard form, so it is necessary to rewrite that equation.

$$\begin{cases} f = m + u \\ f + m + u = 58{,}000 \\ 29f + 19.99m + 13.50u = 1{,}336{,}340 \end{cases} \rightarrow \begin{cases} f - m - u = 0 \\ f + m + u = 58{,}000 \\ 29f + 19.99m + 13.50u = 1{,}336{,}340 \end{cases}$$

Although some of the variables appear to have no coefficients, remember that a coefficient of one is implied. A matrix equation that represents the system above is

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 29 & 19.99 & 13.50 \end{bmatrix}\begin{bmatrix} f \\ m \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 58{,}000 \\ 1{,}336{,}340 \end{bmatrix}.$$

STEP 3 **Represent and solve the system using matrices with technology.**

To represent the system using technology, enter a 3×3 matrix A with entries equal to the coefficient matrix and a 3×1 matrix B with entries equal to the constant matrix.

To solve the system using technology, multiply the inverse of matrix A by matrix B.

$$A^{-1}B = \begin{bmatrix} 29{,}000 \\ 16{,}000 \\ 13{,}000 \end{bmatrix}$$

$\begin{cases} f = m + u \\ f + m + u = 58{,}000 \\ 29f + 19.99m + 13.50u = 1{,}336{,}340 \end{cases}$ ***is the system of linear equations that represents this situation where f is the number of seats in the field level, m is the number of seats in the mezzanine level, and u is the number of seats in the upper level of the baseball stadium.***

$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 29 & 19.99 & 13.50 \end{bmatrix}\begin{bmatrix} f \\ m \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 58{,}000 \\ 1{,}336{,}340 \end{bmatrix}$ ***is the matrix equation that represents the system of equations.***

$\begin{bmatrix} f \\ m \\ u \end{bmatrix} = \begin{bmatrix} 29{,}000 \\ 16{,}000 \\ 13{,}000 \end{bmatrix}$ ***is the solution for the system determined using matrices and technology.***

The baseball stadium has 29,000 seats in its field level, 16,000 seats in its mezzanine level, and 13,000 seats in its upper level.

YOU TRY IT! #2

A snack company sells bulk nuts, fruits, and granola. The company plans to offer a new trail mix for \$4.07 per pound that is composed of granola, peanuts, and raisins. There will be twice as many pounds of peanuts in the new trail mix as raisins. When sold separately, granola sells for \$5.99 per pound, peanuts sell for \$2.99 per pound, and raisins sell for \$3.99 per pound. How many pounds of each ingredient are in 100 pounds of the new trail mix? Write a system of three linear equations to represent the situation and a matrix equation to represent the system. Represent and solve the system using matrices with technology.

PRACTICE/HOMEWORK

For questions 1 – 3, create a matrix equation to represent each system of equations.

1. $\begin{cases} x + 2y - 3z = -2 \\ 2x - 2y + z = 7 \\ 2x + y + 3z = -4 \end{cases}$

2. $\begin{cases} 20a + 9b = 127 \\ 8a + 18b + 3c = 97 \\ 3b + 5c = 14 \end{cases}$

3. $\begin{cases} x + 2y + 3z = 9 \\ x = -2y \\ x + 4y - z = -5 \end{cases}$

Use the situation described below to answer questions 4 – 8.

FINANCE

Carla wants to order gift bags of mixed dried fruits. Three of her options are described in the table below.

DRIED FRUIT BAGS

	DESCRIPTION	COST
MIXED BAG A	3 pounds each of pineapple and strawberries 2 pounds of apples	\$43.00
MIXED BAG B	4 pounds each of pineapple and apples 1 pound of strawberries	\$38.00
MIXED BAG C	5 pounds of pineapples 3 pounds of apples 2 pounds of strawberries	\$46.50

4. Write a system of equations that you could use to represent this situation. Let the variables x, y, and z represent the cost per pound of pineapple, apples, and strawberries, respectively.

5. Use the coefficients of x, y, and z to write matrix A where each row represents one equation and each column represents one of the variables x, y, and z.

6. Use the constants from each equation to write matrix B where each row represents one equation.

7. Use the matrix $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to represent the variables and write a matrix equation relating matrix A, matrix B, and the variable matrix.

8. Solve for $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and explain what each value means for the situation.

Use the situation described below to answer questions 9 – 11.

BUSINESS

A distribution center is sending a shipment of 420 shoes worth a total of \$19,740 to a local store. The shipment contains three types of shoes: Shoe A has a value of \$50, Shoe B has a value of \$65, and Shoe C has a value of \$33.50. There are twice as many of shoe "C" than there are of shoe "B".

9. Write a system of equations that you could use to represent this situation. Let the variables a, b, and c represent the number of each type of shoe (A, B, and C) respectively.

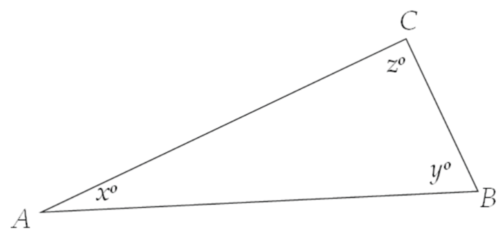
10. Create a matrix equation to represent this system of equations.

11. Solve for $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, and explain what each value means for the situation.

Use the situation described below to answer questions 12 – 14.

GEOMETRY

In the ΔABC, $m\angle C$ is three times $m\angle A$. Also, the sum of $m\angle A$ and $m\angle B$ is equal to $m\angle C$. Remember that in all triangles, the sum of the measures of the interior angles is 180°.



12. Write a system of equations that you could use to represent this situation.

13. Create a matrix equation to represent this system of equations.

14. Solve for $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and explain what each value means for the situation.

Use the situation described below to answer questions 15 – 17.

FINANCE

Raul has an uncle who likes to play games when he visits. On his most recent visit, he told Raul that he could have all the money in his wallet if he were able to guess how many of each kind of bill he had. Here are the clues he gave:

- I have only 3 denominations of bills - \$5 bills, \$10 bills, and \$20 bills.
- I have a total of 17 bills in my wallet.
- The value of all the money in my wallet is \$180.
- I have twice as many \$5 bills as I do \$10 bills.

15. Write a system of equations that you could use to represent this situation. Let the variables a, b, and c represent the number of each type of bill (\$5, \$10, and \$20) respectively.

16. Create a matrix equation to represent this system of equations.

17. Solve for $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, and explain what each value means for the situation.

Use the situation described below to answer questions 18 – 20.

FINANCE

A school's Theater Club is performing a play for the general public. The table below shows the cost of each type of ticket the Theater Club will sell.

TICKET PRICES	
CHILD (UP TO 11 YEARS)	$2.50
STUDENT (12 – 18 YEARS)	$5.00
ADULT (19 YEARS AND UP)	$7.50

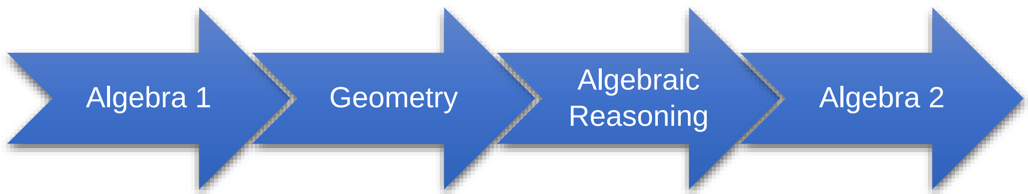
They sold 600 tickets and took in $3,085. The number of Student tickets sold was twice the sum of Child and Adult tickets.

18. Write a system of equations that you could use to represent the number of each type of ticket. Let the variables c, s, and a represent the number of each type of ticket (child, student, adult) respectively. Rewrite the system so that all equations are in standard form.

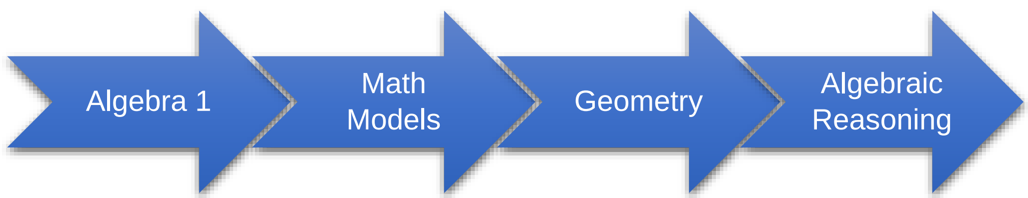
19. Create a matrix equation to represent this system of equations.

20. Solve the matrix equation, and explain what each value means for the situation.

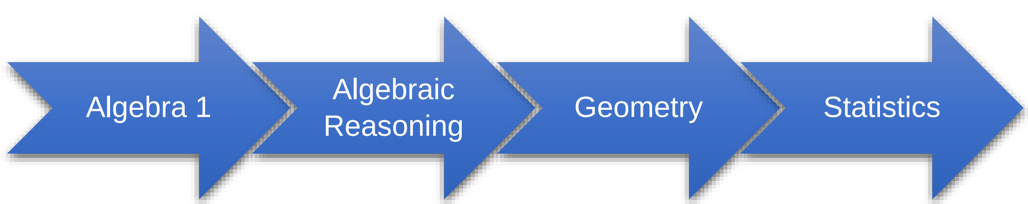
Trinity High School uses Algebraic Reasoning to support students who need additional help with their Algebra 1 skills before they enter Algebra 2. Many students take the traditional Algebra 1-Geometry-Algebra 2-Precalculus pathway. The district has locally defined Algebra 2 as a graduation requirement. Yet, counselors know that many students may not be successful if they enter Algebra 2 directly.



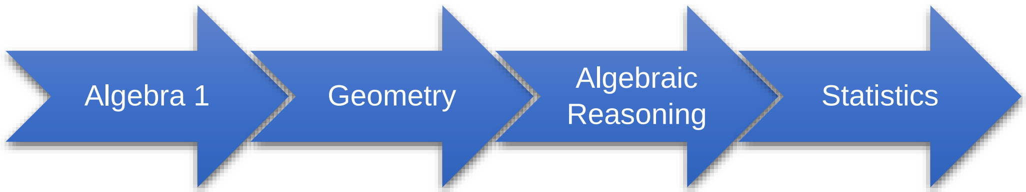
Sabine High School uses Algebraic Reasoning as an alternative to Algebra 2 for students who had a difficult time in Algebra 1. Students who struggle in Algebra 1 take Math Models, frequently while they retake the Algebra 1 EOC. Then, they take Geometry as juniors so that if they fail Geometry, they have one more chance to take Geometry again as a senior and at least graduate with a Foundation diploma. Most students take the traditional Algebra 1-Geometry-Algebra 2 course sequence, so this course pathway is reserved for students who are most likely to struggle in Algebra 2.



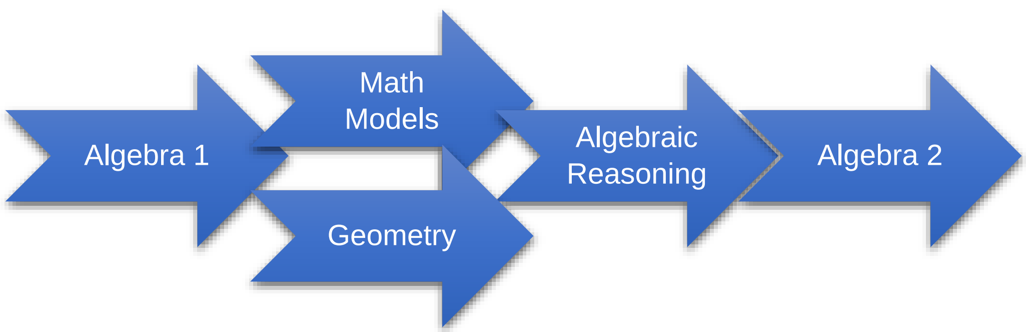
Red River High School uses Algebraic Reasoning for students who do not earn a grade of at least 75 in Algebra 1 or who do not pass the Algebra 1 EOC. Algebraic Reasoning curriculum is taught, but students who need to pass the Algebra 1 EOC are pulled out for EOC-specific tutorials. Counselors feel that these students will likely not pursue a career in a STEM field after graduation, so they then take Geometry as juniors and Statistics as seniors. They can still earn any endorsement other than STEM and will not have the pressure of taking Algebra 2.



Brazos High School uses Algebraic Reasoning as an alternative to Algebra 2. Students who take Algebraic Reasoning typically want to obtain an endorsement but do not want to take Algebra 2, which is reserved for calculus-intending students. After taking Algebraic Reasoning, students take Statistics so that they will have an endorsement (other than STEM) and be prepared to attend college or university.



Colorado High School double-blocks Geometry Math Models (MMA) for students who do not pass the Algebra 1 EOC but passed the Algebra 1 course. Then, students take Algebraic Reasoning as 11th graders to prepare them for Algebra 2 as 12th graders. Having Algebraic Reasoning in place for 11th grade also provides students who are still retaking the Algebra 1 EOC a focus on algebraic skills before taking Algebra 2.



Neches High School made an innovative partnership with their local community college. Instead of having Algebra 2 as a prerequisite for Dual Credit College Algebra, some students take Algebraic Reasoning in lieu of Algebra 2 in order to provide a more solid conceptual foundation for advanced algebra skills. Then, students enroll in Dual Credit College Algebra in order to learn advanced algebra that would be addressed in an Algebra 2 class while earning college credit.